Core Books in Advanced Mathematics

Mechanics of Groups of Particles

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Preface

Advanced level mathematics syllabuses are once again undergoing changes in content and approach following the revolution in the early 1960s which led to the unfortunate dichotomy between 'modern' and 'traditional' mathematics. The current trend in syllabuses for Advanced level mathematics now being developed and published by many GCE Boards is towards an integrated approach, taking the best of the topics and approaches of modern and traditional mathematics, in an attempt to create a realistic examination target through syllabuses which are maximal for examining and minimal for teaching. In addition, resulting from a number of initiatives, core syllabuses are being developed for Advanced level mathematics consisting of techniques of pure mathematics as taught in schools and colleges at this level.

The concept of a core can be used in several ways, one of which is mentioned above, namely the idea of a core syllabus to which options such as theoretical mechanics, further pure mathematics and statistics can be added. The books in this series involve a different use of the core idea. They are books on a range of topics, each of which is central to the study of Advanced level mathematics; they form small core studies of their own, of topics which together cover the main areas of any single-subject mathematics syllabus at Advanced level.

Particularly at times when economic conditions make the problems of acquiring comprehensive textbooks giving complete syllabus coverage acute, schools and colleges and individual students can collect as many of the core books as they need to supplement books they already have, so that the most recent syllabuses of, for example, the London, Cambridge, AEB and JMB GCE Boards can be covered at minimum expense. Alternatively, of course, the whole set of core books gives complete syllabus coverage of single-subject Advanced level mathematics syllabuses.

The aim of each book is to develop a major topic of the single-subject syllabuses giving essential book work, worked examples and numerous exercises arising from the authors' vast experience of examining at this level. Thus, as well as using the core books in either of the above ways, they are ideal for supplementing comprehensive textbooks by providing more examples and exercises, so necessary for the preparation and revision for examinations.

A feature of the books in this series is the large number of worked examples which are regarded as an integral part of the text. It is hoped that these books will be of use not only to schools but also to the large minority of candidates who are studying by themselves. Mechanics, perhaps more so than other

branches of mathematics, can be mastered only by working conscientiously and thoughtfully through many worked and unworked examples. This is the only way in which the essential simplicity of the subject can be appreciated.

In order to achieve a degree of uniformity in the size of the books in this series, the Advanced level mechanics topics are covered in two books. The first book, *Newton's Laws and Particle Motion*, considered the basic principles of mechanics and applied them to particle motion. In this book basic kinematics and Newton's laws, as covered in the first book, are used to discuss energy, groups of particles, statics and rigid-body motion.

The treatment in both books assumes a knowledge of vector algebra and of elementary calculus (including the exponential and logarithm functions). These subjects are covered in other volumes in the series. The worked examples and problems have been carefully chosen to be physically important (as distinct from artificial examples devoid of any relevance to the real world). The treatment unashamedly makes appeals to intuition and common-sense and the ordering of the topics (sometimes unconventional) has been arranged to stress that all mechanics is based on Newton's laws, not on a series of *ad hoc* techniques. One final feature is the stress on dimensional consistency and the possibility of checking that this allows.

Whilst the notation used is generally self-explanatory, it should be noted that there are two distinct ways of specifying the inverse trigonometric functions. Thus $\tan^{-1} x$ and $\arctan x$ are both used to denote the angle θ between $-\pi/2$ and $\pi/2$ such that $\tan\theta = x$. Also, the symbol $\approx$ means 'approximately equals'.

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1 Energy, work and power

1.1 Introduction

In order to use mathematics efficiently to deal with the mechanics of groups of particles, including rigid bodies, it is helpful first to use Newton's laws to derive other principles. In this chapter, Newton's Second Law is used to introduce the concepts of kinetic energy and work, which are illustrated for several different types of motion. This chapter also contains a brief introduction to the important idea of the power of a force.

1.2 Work and energy for a particle moving in a straight line

Consider a particle of mass m moving along a straight line in the direction of the unit vector $\mathbf{i}$ under the influence of a force $\mathbf{F} = F\mathbf{i}$. Let the velocity of the particle be $\mathbf{v} = v\mathbf{i}$. By Newton's Second Law

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} \quad \Rightarrow \quad m\frac{\mathrm{d}v}{\mathrm{d}t} = F. \tag{1.1}$$

This can be written

$$m\frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}v^2) = F$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}mv^2) = F, \tag{1.2}$$

since the mass m of the particle can be regarded as constant. In equations (1.1) and (1.2) v and F can be positive or negative. The important quantity $\frac{1}{2}mv^2$ appearing on the left-hand side of (1.2) is known as the *kinetic energy* of the particle.

We now restrict our consideration to situations in which F is a function of x. This includes the following special cases:

(i) F is constant;
(ii) F is proportional to x;
(iii) F is a linear function of x, as in simple harmonic motion.

Suppose that the particle moves from a position where $x = x_1$ and $v = v_1$ to a position where $x = x_2$ and $v = v_2$. We can then integrate (1.2) with respect to x from $x = x_1$ to $x = x_2$ and obtain

$$\left[\tfrac{1}{2}mv^2\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} F\,\mathrm{d}x$$

$$\Rightarrow \quad \tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2} = \int_{x_1}^{x_2} F\,\mathrm{d}x. \tag{1.3}$$

The right-hand side of (1.3), namely

$$\int_{x_1}^{x_2} F\,\mathrm{d}x,$$

is the work done by the force as x changes from x_1 to x_2. Therefore (1.3) can be stated:

'change in kinetic energy = work done'.

Hence both kinetic energy and work done have the same dimensions $\mathrm{ML^2T^{-2}}$ and the same units $\mathrm{kgm^2s^{-2}}$. Kinetic energy and work are important quantities and the special name of 1 joule (abbreviated to 1 J) is given to the quantity $1\ \mathrm{kgm^2s^{-2}}$. Thus 1 J is the kinetic energy of a particle of mass 1 kg moving with a speed of $1\ \mathrm{ms^{-1}}$.

We now consider the special cases of (1.3) as listed earlier.

(i) Suppose that F is constant. Equation (1.3) then becomes

$$\tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2} = Fx_2 - Fx_1. \tag{1.4}$$

As a first application consider the particle to be moving vertically so that the force F is its own weight. Choosing $\mathbf{i}$ to be vertically upwards, we have $F = -mg$. Then (1.4) becomes

$$\tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2} = -mgx_2 + mgx_1. \tag{1.5}$$

Suppose that the particle has been released from rest and after falling a distance h has a speed V. We can then write $x_1 = v_1 = 0$, $x_2 = -h$, $v_2 = V$ in (1.5) and obtain

$$\tfrac{1}{2}mV^2 = mgh. \tag{1.6}$$

In falling through a vertical distance h, the work done by the weight of the particle is mgh, which has been converted into kinetic energy to the amount $\tfrac{1}{2}mV^2$.

Another way of interpreting (1.6) is useful both here and in many other applications. Consider a particle of mass m held at rest a height h above the Earth's surface. If the particle were released from rest, the Earth would do work on it as it fell and the particle would acquire kinetic energy of an amount equal to the work done on it. Therefore we can regard the particle held at rest above the Earth's surface as having *potential energy*, that is energy that can be converted into kinetic energy. We can define the potential energy of the particle of mass m held at height h above the Earth's surface as the work that would be

done on it by the Earth's gravity if it were allowed to fall to the surface, that is mgh. More generally, we now see that (1.5) can be stated as:

$$\text{'increase (or decrease) in kinetic energy} \\ = \text{decrease (or increase) in potential energy'.} \tag{1.7}$$

Also (1.5) can be rearranged in the form

$$\tfrac{1}{2}mv_1{}^2 + mgx_1 = \tfrac{1}{2}mv_2{}^2 + mgx_2. \tag{1.8}$$

If the Earth's surface is taken as $x = 0$, then the left-hand side of this equation is the sum of the kinetic energy and potential energy of the particle when $x = x_1$, and the right-hand side is the sum when $x = x_2$. Since x_1 and x_2 are any values of x, (1.8) states that during the motion

$$\text{'kinetic energy + potential energy = constant'.} \tag{1.9}$$

Note that (1.9) remains true whatever position is chosen as $x = 0$. For suppose the Earth's surface is at $x = x_0$. Then the potential energies of the particle when $x = x_1$ and $x = x_2$ are $mg(x_1 - x_0)$ and $mg(x_2 - x_0)$ respectively. By adding $-mgx_0$ to both sides of (1.8) we recover equation (1.9). The point is that the physical effects of potential energy depend only on the changes in potential energy during motion and not on the position where the potential energy is defined to be zero.

Equation (1.9) is a statement of the *principle of conservation of energy*, other examples of which we shall meet later.

Example 1 When the brakes are applied, the resistance to the motion of a car of mass m is constant and equal to $0{\cdot}8\ mg$. Given that the speed of the car just before the brakes are applied is 50 kmh^{-1}, estimate the distance in which it comes to rest, giving your answer in metres.

$$1\ \mathrm{h} = 3600\ \mathrm{s} \quad \Rightarrow \quad 50\ \mathrm{km\,h}^{-1} \approx 13{\cdot}89\ \mathrm{ms}^{-1}.$$

Since the resistance is constant, we can use (1.4) with $F = -0{\cdot}8mg$; the minus sign is inserted because the resistance opposes the motion. We have $v_1 \approx 13{\cdot}89\ \mathrm{ms}^{-1}$ and $v_2 = 0$.

$$(1.4) \quad \Rightarrow \quad -\tfrac{1}{2}m(13{\cdot}89)2 = -(0{\cdot}8)m(9{\cdot}81)s,$$

where s m $= x_2 - x_1$ is the required distance [We have put $g = 9{\cdot}81\ \mathrm{ms}^{-2}$.]

$$\Rightarrow \quad s = (13{\cdot}89)^2/(1{\cdot}6 \times 9{\cdot}81) \approx 12{\cdot}3.$$

(ii) Now let us consider (1.3) when F is not constant but varies with x. As a first example, suppose that

$$F = -\frac{GMm}{x^2},$$

where G and M are constants. (This is Newton's Law of Gravitation. If M is the mass of the Earth, the attractive force on a particle of mass m, for example

the Moon, is of this form, where x is the distance of the particle from the centre of the Earth.)

$$(1.3) \quad \Rightarrow \quad \tfrac{1}{2}mv_2{}^2 - \tfrac{1}{2}mv_1{}^2 = -\int_{x_1}^{x_2} \frac{GMm}{x^2}\mathrm{d}x = \frac{GMm}{x_2} - \frac{GMm}{x_1}.$$

We note that the equation can be rearranged to give

$$\tfrac{1}{2}mv_1{}^2 + \left(-\frac{GMm}{x_1}\right) = \tfrac{1}{2}mv_2{}^2 + \left(-\frac{GMm}{x_2}\right). \tag{1.10}$$

In this form the result can be interpreted in terms of energy. Since x_1 and x_2 are any values of x, (1.10) shows that, throughout the motion,

$$\tfrac{1}{2}mv^2 + \left(-\frac{GMm}{x}\right) = \text{constant}. \tag{1.11}$$

We recognize $\frac{1}{2}mv^2$ as the kinetic energy of the particle, so that (1.11) would be consistent with (1.9), the principle of conservation of energy, if it were possible to identify $(-GMm/x)$ as the potential energy of the particle.

Let us denote the potential energy by Φ.

From (1.9),
$$\tfrac{1}{2}mv^2 + \Phi = \text{constant}.$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}mv^2) + \frac{\mathrm{d}}{\mathrm{d}x}(\Phi) = 0.$$

From (1.2),
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}mv^2) = F,$$

$$\Rightarrow \quad \frac{\mathrm{d}\Phi}{\mathrm{d}x} = -F. \tag{1.12}$$

We adopt (1.12) as a *definition* of Φ for any motion in a straight line for which F depends only on x, and we recall (from an earlier comment) that only *changes* in Φ are physically important. Therefore we can choose for Φ *any* quantity satisfying (1.12). It follows that we can take $\Phi = (-GMm/x)$ when

$$F = (-GMm/x^2)$$

since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{GMm}{x}\right) = \frac{GMm}{x^2}.$$

We now see that equations (1.10) and (1.11) are consistent with the principle of conservation of energy.

(iii) As a final illustration of (1.3), we consider S.H.M. (simple harmonic motion). However, this is so important that we have devoted an entire section to it.

1.3 Energy considerations for simple harmonic motion

Let us consider a particle of mass m executing S.H.M. with centre $x = 0$, amplitude A and period $(2\pi/n)$. From the definition of S.H.M. it follows that the displacement x of the particle at time t satisfies the equation

$$x = A\sin(nt + \phi), \tag{1.13}$$

where ϕ is a constant. Then the speed $v = \dot{x}$ of the particle at time t satisfies

$$v = nA\cos(nt + \phi)$$

$$\Rightarrow \quad \tfrac{1}{2}mv^2 = \tfrac{1}{2}mn^2A^2\cos^2(nt + \phi).$$

From (1.13), $x^2 = A^2\sin^2(nt + \phi)$. Hence $A^2\cos^2(nt + \phi) = A^2 - x^2$

$$\Rightarrow \quad \tfrac{1}{2}mv^2 + \tfrac{1}{2}mn^2x^2 = \tfrac{1}{2}mn^2A^2. \tag{1.14}$$

We now show that (1.14) is consistent with (1.9), the principle of conservation of energy. To do this we have to find first F, and then Φ.

Newton's Second Law $\Rightarrow$ $F = m\dot{v} = -mn^2A\sin(nt + \phi) = -mn^2x.$

$$(1.12) \quad \Rightarrow \quad \frac{\mathrm{d}\Phi}{\mathrm{d}x} = mn^2x$$

$$\Rightarrow \quad \Phi = \tfrac{1}{2}mn^2x^2. \tag{1.15}$$

Thus (1.14) is consistent with the principle of conservation of energy.

Example 2 A particle is performing S.H.M. with centre O. When the particle is at B, where $OB = 1$ m, its speed is $3\sqrt{3}$ ms^{-1}, and when it is at C (on the same side of O as B), where $OC = \sqrt{3}$ m, its speed is 3 ms^{-1}. Find the maximum distance from O reached by the particle, and show that it takes about 0·175 s to travel directly from B to C.

Let the amplitude of the motion be A m, and let $(2\pi/n)$ s be its period. We now insert the data into (1.14).

$$x = 1\text{ m}, v = 3\sqrt{3}\text{ ms}^{-1} \quad \Rightarrow \quad 27 + n^2 = n^2A^2;$$

$$x = \sqrt{3}\text{ m}, v = 3\text{ ms}^{-1} \quad \Rightarrow \quad 9 + 3n^2 = n^2A^2$$

$$\Rightarrow \quad 27 + n^2 = 9 + 3n^2 \quad \Rightarrow \quad n^2 = 9 \quad \Rightarrow \quad n = 3 \quad (\text{since } n \text{ is positive})$$

$$\Rightarrow \quad 9A^2 = 27 + 9 \quad \Rightarrow \quad A^2 = 4 \quad \Rightarrow \quad A = 2.$$

Hence the amplitude of the motion, that is its maximum distance from O, is 2 m. Let the particle be at B and C at times t_1 s and t_2 s respectively. We use the data, and the values of n and A, in (1.13).

$$x = 1\text{ m}, t = t_1\text{ s} \quad \Rightarrow \quad 1 = 2\sin(3t_1 + \phi) \quad \Rightarrow \quad 3t_1 + \phi = \pi/6;$$

$$x = \sqrt{3}\text{ m}, t = t_2\text{ s} \quad \Rightarrow \quad \sqrt{3} = 2\sin(3t_2 + \phi) \quad \Rightarrow \quad 3t_2 + \phi = \pi/3.$$

$$\Rightarrow \quad t_2 - t_1 = \frac{1}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{18} \approx 0{\cdot}175.$$

The methods used above can be applied to S.H.M. with centre $x = x_0 \neq 0$. Equation (1.13) is then replaced by

$$x = x_0 + A \sin(nt + \phi),$$

and it is left as an exercise for the reader to show that the potential energy Φ is now $\frac{1}{2}mn^2(x - x_0)^2$, and that (1.14) becomes

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}mn^2(x - x_0)^2 = \tfrac{1}{2}mn^2A^2. \tag{1.16}$$

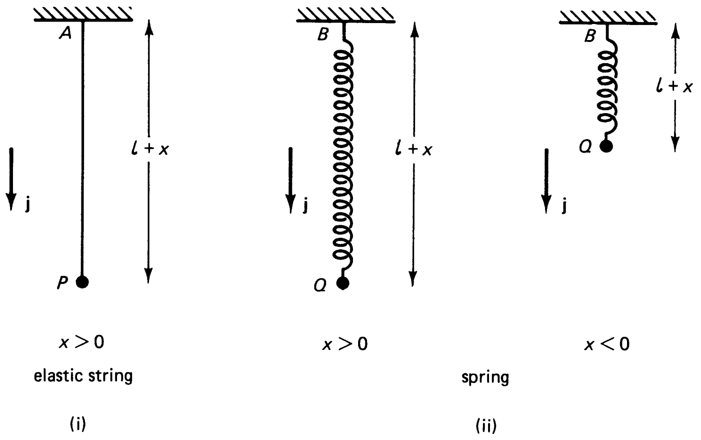


Fig. 1.1

Let us now consider the situations shown in Fig. 1.1. A particle of mass m is attached to one end of an elastic string of natural length l (Fig. 1.1(i)) or a spring of natural length l (Fig. 1.1(ii)). The other end of the string (or spring) is fastened to a fixed point. In equilibrium the particle is vertically below the fixed point and the extension of the string (or spring) is given by mgl/λ, where λ is the modulus of elasticity. Motion of the particle in a vertical line can be caused by giving it a vertical blow, or by pulling it down and then releasing it. Let the extension of the string (or spring) in the subsequent motion be x. Then (provided that $x > 0$ for all t in the case of an elastic string, Fig. 1.1(i)), Newton's Second Law and Hooke's Law give the equation for x as

$$m\ddot{x} = mg - (\lambda x/l).$$

This can be rewritten as

$$m\frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}v^2) = F, \quad \text{where } F = mg - \lambda x/l. \tag{1.17}$$

There are therefore two contributions to F, one from the weight of the particle and the other from the restoring force in the string (or spring). According to (1.12) it follows that the potential energy Φ satisfies

$$\frac{d\Phi}{dx} = -mg + \frac{\lambda x}{l}$$

$$\Rightarrow \quad \Phi = -mgx + \lambda x^2/2l. \tag{1.18}$$

The term $-mgx$ is the potential energy associated with the particle's weight (notice that the minus sign occurs here because x increases in a downwards direction rather than upwards as in the discussion in §1.2). The term $\lambda x^2/2l$ is the potential energy associated with the elastic string (or spring); we can regard an elastic string (or spring) of length $l + x$ as having an energy $\lambda x^2/2l$ stored in it which can be converted into kinetic energy.

Substituting (1.18) into the expression for the conservation of energy (1.9) gives

$$\tfrac{1}{2}mv^2 - mgx + \lambda x^2/2l = C, \tag{1.19}$$

where C is a constant. Putting $n^2 = \lambda/ml$ and $x_0 = mgl/\lambda$, we obtain

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}mn^2x^2 - mn^2x_0x = C$$

$$\Rightarrow \quad \tfrac{1}{2}mv^2 + \tfrac{1}{2}mn^2(x - x_0)^2 = C + \tfrac{1}{2}mn^2{x_0}^2,$$

which is (1.16) when we identify the constant C as $\tfrac{1}{2}mn^2(A^2 - {x_0}^2)$.

Example 3 One end of an elastic string of natural length l is fastened to a particle of mass m and the other end is fastened to a fixed point A. The particle is held at A and released, when it falls vertically. The greatest length of the string subsequently is $2l$. Show that the modulus of elasticity of the string is $4mg$.

Show that the time between the string first becoming taut and first becoming slack again is approximately $1{\cdot}91\sqrt{(l/g)}$.

Let x denote the extension of the string. When the particle has fallen through a distance l the string is just taut and $x = 0$, so there is no energy stored in the string. Hence the downwards speed V of the particle is then $\sqrt{(2gl)}$, given by (1.6) with $h = l$. For $x > 0$ we use the principle of conservation of energy in the form (1.19):

$$\tfrac{1}{2}mv^2 - mgx + \lambda x^2/2l = C.$$

$$v = \sqrt{(2gl)} \text{ when } x = 0 \quad \Rightarrow \quad C = \tfrac{1}{2}mV^2 = mgl$$

$$v = 0 \text{ when } x = l \quad \Rightarrow \quad -mgl + \lambda l/2 = mgl$$

$$\Rightarrow \quad \lambda = 4mg.$$

When $x > 0$ the particle performs S.H.M. with the values of n^2 and x_0 given by

$$n^2 = \lambda/ml = 4g/l, \quad x_0 = mgl/\lambda = l/4.$$

Hence
$$x = x_0 + A\sin(nt + \phi).$$

Let us measure t from the first instant when $x = 0$; we require the value of t when $x = 0$ for the *second* time.
When $x = 0$, $t = 0$

$$-x_0 = A \sin \phi \quad \Rightarrow \quad A \sin \phi = -l/4.$$

Also when $t = 0$

$$\dot{x} = V = \sqrt{(2gl)} \quad \Rightarrow \quad \sqrt{(2gl)} = An \cos \phi \quad \Rightarrow \quad A \cos \phi = l/\sqrt{2}$$

$$\Rightarrow \quad A^2 = A^2 \sin^2 \phi + A^2 \cos^2 \phi = \frac{l^2}{16} + \frac{l^2}{2} = \frac{9l^2}{16}$$

$$\Rightarrow \quad A = \frac{3l}{4}$$

$$\Rightarrow \quad \sin \phi = -\frac{1}{3}, \quad \cos \phi = (2\sqrt{2})/3$$

$$\Rightarrow \quad -\frac{\pi}{2} < \phi < 0, \tan \phi = -1/(2\sqrt{2})$$

$$\Rightarrow \quad \phi \approx -0{\cdot}3398 \text{ rad.}$$

When $x = 0$

$$\sin (nt + \phi) = -\frac{x_0}{A} = -\frac{1}{3} = \sin \phi.$$

This equation is (of course) satisfied by $t = 0$, which is when the string first becomes taut. We want the next solution (when the particle reaches $x = 0$ travelling upwards) which occurs when $nt + \phi = \pi - \phi$.

$$\Rightarrow \quad t = n^{-1}(\pi - 2\phi) \approx 3{\cdot}821n^{-1} = 3{\cdot}821(4g/l)^{-1/2} \approx 1{\cdot}91\sqrt{(l/g)}.$$

1.4 Motion of a particle in a vertical circle

The principle of conservation of energy, so far considered only for the case of the motion of a particle in a straight line under the action of a force which depends only on the particle's displacement, has much wider application. This statement is well demonstrated by considering the motion shown in Fig. 1.2.

A particle P of mass m is attached to one end of a light, inelastic string of length l. The other end of the string is fastened to a fixed point O. In equilibrium P is at rest vertically below O. Suppose P is then given a horizontal blow so that motion occurs. Because the string is inextensible P begins to move on a vertical circle with centre O and radius l. Suppose OP makes an angle θ with the downward vertical at time t. Let the velocity and acceleration of P at time t be $\mathbf{v}$ and $\mathbf{a}$ respectively. From the kinematics of circular motion, we have

$$\mathbf{v} = l\dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \mathbf{a} = l\ddot{\theta}\hat{\boldsymbol{\theta}} - l\dot{\theta}^2\hat{\mathbf{r}}. \tag{1.20}$$

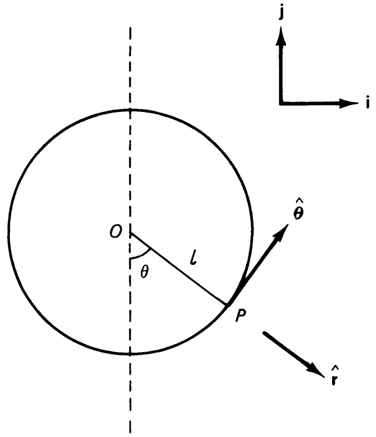


Fig. 1.2

In (1.20) $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the directions shown in Fig. 1.2.

The particle is acted on by two forces, its weight $-mg\mathbf{j}$ and the tension in the string $-T\hat{\mathbf{r}}$. Since $\mathbf{j} = \hat{\boldsymbol{\theta}}\sin\theta - \hat{\mathbf{r}}\cos\theta$, the total force $\mathbf{F}$ on P is given by

$$\mathbf{F} = -\hat{\boldsymbol{\theta}}mg\sin\theta + \hat{\mathbf{r}}(mg\cos\theta - T). \tag{1.21}$$

Newton's Second Law $\Rightarrow$ $-\hat{\boldsymbol{\theta}}mg\sin\theta + \hat{\mathbf{r}}(mg\cos\theta - T) = ml\ddot{\theta}\hat{\boldsymbol{\theta}} - ml\dot{\theta}^2\hat{\mathbf{r}}$.

$$\Rightarrow \quad l\ddot{\theta} = -g\sin\theta, \quad T = mg\cos\theta + ml\dot{\theta}^2. \tag{1.22}$$

The first equation in (1.22) can, in principle, be solved to find θ in terms of t; then the value of T can be found from the second equation in (1.22).

In fact we can obtain the value of T without completely solving the first equation in (1.22). From the first equation in (1.20) we note that the speed v of P satisfies

$$v = |\mathbf{v}| = l\dot{\theta}.$$

Hence $l\ddot{\theta} = \dot{v}$. However by using the function of a function rule (or chain rule) of differentiation we have

$$l\ddot{\theta} = \dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{l}v\frac{\mathrm{d}v}{\mathrm{d}\theta} = \frac{1}{2l}\frac{\mathrm{d}}{\mathrm{d}\theta}(v^2).$$

Hence the first equation in (1.22) can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\tfrac{1}{2}mv^2) = -mgl\sin\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}(mgl\cos\theta).$$

Thus $\frac{1}{2}mv^2$ and $mgl\cos\theta$, having the same derivative, differ by a constant,

$$\Rightarrow \quad \tfrac{1}{2}mv^2 - mgl\cos\theta = \text{constant}. \tag{1.23}$$

In (1.23) we recognize $\frac{1}{2}mv^2$ as the kinetic energy of P. But we also see that $l\cos\theta$ is the vertical distance of P below the level of O (or, equivalently, $-l\cos\theta$ is the vertical distance of P above the level of O). Hence, according to the discussion in §1.2, $-mgl\cos\theta$ is the potential energy of P due to its weight. Therefore (1.23) is, once more, a statement of (1.9), the principle of conservation of energy.

However there is no contribution to the energy equation (1.23) from the tension in the string $-T\hat{\mathbf{r}}$. We shall see later in §1.7 that this is because $-T\hat{\mathbf{r}}$ is perpendicular to $\mathbf{v}$, since $\hat{\mathbf{r}} . \hat{\boldsymbol{\theta}} = 0$.

The following example illustrates further aspects of the motion of a particle in a vertical circle.

Example 4 Given that when $\theta = 0$, in the motion shown in Fig. 1.2, the speed of P is $\sqrt{(\alpha gl)}$, where α is a constant, find the magnitude T of the tension in the string. Hence determine the values of α for which P performs complete circles. Describe very briefly the motion of P when (i) $\alpha = 3$, (ii) $\alpha = 2$.

From information in the first sentence, the constant in (1.23) can be determined. When $\theta = 0$, $\cos\theta = 1$ and $v = \sqrt{(\alpha gl)}$

$$\Rightarrow \quad \tfrac{1}{2}m\alpha gl - mgl = \text{constant},$$

$$\Rightarrow \quad v^2 = 2gl[\cos\theta + \tfrac{1}{2}(\alpha - 2)]. \tag{1.24}$$

Since $v = l\dot{\theta}$, the second equation of (1.22) gives

$$T = mg\cos\theta + \frac{mv^2}{l} = 3mg\left[\cos\theta + \frac{1}{3}(\alpha - 2)\right]. \tag{1.25}$$

In deriving (1.24) and (1.25) we have assumed that P moves in a circular path, that is that $OP = l$ for all θ. However, since O and P are connected by a string, T cannot be negative. From (1.25) it follows that θ must satisfy $\cos\theta + \frac{1}{3}(\alpha - 2) \geqslant 0$, and since the minimum value of $\cos\theta$ is -1, occurring when $\theta = 180°$ (i.e. when P is vertically above O), complete revolutions can occur only if

$$-1 + \tfrac{1}{3}(\alpha - 2) \geqslant 0 \quad \Rightarrow \quad \alpha \geqslant 5. \tag{1.26}$$

Furthermore, when $\alpha \geqslant 5$, equation (1.24) shows that $v^2 > 0$ for all θ. Hence (1.26) states the conditions for complete revolutions to occur.

(i) For $\alpha = 3$, $v^2 = 2gl[\cos\theta + \frac{1}{2}]$ and $T = 3mg[\cos\theta + \frac{1}{3}]$.

In this case T becomes zero, so that the string becomes slack when $\cos\theta = -\frac{1}{3}$, that is $\theta \approx 109{\cdot}5°$. For this value of $\cos\theta$, $v^2 = gl/3$. Hence P moves round the circle until $\theta \approx 109{\cdot}5°$, after which it moves in a parabolic path as a projectile under the action of its weight alone. The situation is illustrated in Fig. 1.3(i). Eventually, of course, the string becomes taut again.

(ii) For $\alpha = 2$, $v^2 = 2gl\cos\theta$ and $T = 3mg\cos\theta$.

In this case T and v become zero when θ is 90°, that is when OP is horizontal.

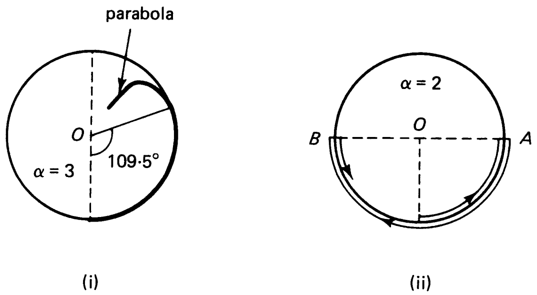


Fig. 1.3

So when the string is just on the point of becoming slack, the speed of P is also on the point of vanishing. Thus, when P reaches A, shown in Fig. 1.3(ii), it stops and then returns along the path it has just taken. It continues moving until T and v are again zero, which happens when $\theta = -90°$. Thus P oscillates between A and B.

Motion in a vertical circle can take place in other ways. For example P could be joined to O by a light, rigid rod free to turn smoothly about a horizontal axis through O, or P could be a bead threaded onto a smooth hoop in a vertical plane. In these cases we can still derive (1.21) and (1.22) in which T is the magnitude of the force (acting along PO) exerted on P by the rod or the hoop. However, in these two cases, T is not restricted to be non-negative. Accordingly, for the motion considered in Example 4, the condition for complete revolutions to occur is not obtained from the equation for T but from the requirement that $v^2 \geqslant 0$ for all θ. The reader should show that this requires $\alpha \geqslant 4$.

1.5 The simple pendulum

There is an important and interesting special case of the motion considered in §1.4. Suppose once more that the motion is generated by giving P a horizontal blow when it is hanging vertically below O, but suppose that the blow is so weak that θ, the angle made by OP with the downward vertical, is always *small.* When θ is small, $\sin\theta \approx \theta$. Hence the first equation in (1.22) can be approximated by

$$\ddot{\theta} = -n^2\theta, \quad \text{where } n^2 = g/l. \tag{1.27}$$

Apart from the replacement of x by θ, this is the equation of S.H.M. Consequently the general solution of (1.27)—see equation (1.13)—can be written as

$$\theta = A\sin(nt + \phi), \tag{1.28}$$

where A and ϕ are constants. The values of A and ϕ are determined by the initial conditions. Here we are given that $\theta = 0$ when $t = 0$

$$\Rightarrow \quad A\sin\phi = 0 \quad \Rightarrow \quad \phi = 0.$$

Also, the initial speed of P is determined by the magnitude of the blow given to it. By differentiating (1.28) we obtain $v = l\dot{\theta} = lnA\cos(nt + \phi)$. Let the initial speed of P be V. Then, since $\phi = 0$ and $n = \sqrt{(g/l)}$,

$$V = lA\sqrt{(g/l)} \quad \Rightarrow \quad A = V/\sqrt{(lg)}. \tag{1.29}$$

Hence, in this particular case

$$\theta = \theta_0 \sin nt \quad \text{where } \theta_0 = V/\sqrt{(lg)}. \tag{1.30}$$

Equation (1.30) shows that θ oscillates between $\pm\theta_0$. Hence we refer to θ_0 as the *amplitude* of the motion. One complete *oscillation* (e.g. the interval between occasions when $\theta = \theta_0$) takes a time T to complete, where

$$T = 2\pi/n = 2\pi\sqrt{(l/g)}. \tag{1.31}$$

T is the *period* of the motion. The system studied here is called a *simple pendulum*.

These results have been derived on the assumption that θ is always small. Since θ_0 is the maximum value of θ, our assumption is justified provided $\theta_0 = V/\sqrt{(lg)}$ is much less than 1, that is provided the initial speed of P is much less than $\sqrt{(gl)}$.

Example 5 Estimate the length l of a 'seconds' pendulum at a place on the Earth's surface where $g = 9{\cdot}81$ ms^{-2}.

A 'seconds' pendulum is a simple pendulum for which $\frac{1}{2}T = 1$ s. From (1.31) $l = gT^2/4\pi^2$.

$$g = 9{\cdot}81 \text{ ms}^{-2} \quad \text{and} \quad \tfrac{1}{2}T = 1 \text{ s} \quad \Rightarrow \quad l \approx 0{\cdot}994 \text{ m}.$$

1.6 Power for a particle moving in a straight line

We return to the problem considered in §1.2, namely a particle of mass m moving along a straight line with velocity $v\mathbf{i}$ under the influence of a force $F\mathbf{i}$. From (1.1)

$$m\frac{dv}{dt} = F.$$

Multiply both sides of this equation by v. Since $mv\frac{dv}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2)$ we have

$$\frac{d}{dt}(\tfrac{1}{2}mv^2) = Fv. \tag{1.32}$$

Hence Fv is the rate of increase with respect to time of the kinetic energy of the particle. We call Fv the *power*. Noting that, by definition,

$$Fv = \lim_{\delta t \to 0}\left[F\frac{\delta x}{\delta t}\right] = \lim_{\delta t \to 0}\left[\frac{F\delta x}{\delta t}\right],$$

we see that the power is (the limit of) the work, $F\delta x$, done by the force in the time interval δt, divided by δt. Hence power is the *rate of doing work*.

The units of power are kg m^{2}s^{-3} and the special name of 1 watt (abbreviated to 1 W) is given to the quantity of 1 kg m^{2}s^{-3}. Since 1 J $=$ 1 kg m^{2}s^{-2}, we note that 1 W $=$ 1 Js^{-1}. For many practical purposes 1 W is too small a unit, and it is common to express power in terms of kW (the kilowatt, with 1 kW $= 10^3$ W) or even MW (the megawatt, with 1 MW $= 10^6$ W).

We have already seen the importance of the work done by a force. However it is sometimes important to know not only how much work is done by a force, but also how quickly this work can be done. It is in such cases that the concept of power is essential.

Example 6 Estimate the power of a pump which delivers 6 m^3 of water per minute to a height of 20 m through a pipe of cross-sectional area 0·004 m^2. Assume that $g = 9{\cdot}81$ ms^{-2} and that 1 m^3 of water has mass 10^3 kg.

The pump is assumed to have constant power. In each minute it gives both potential energy and kinetic energy to 6 m^3 of water. The potential energy given per minute is

$$6 \times 10^3 \times 20 \times 9{\cdot}81 \text{ J} = 1{\cdot}1772 \times 10^6 \text{ J}.$$

To deliver 6 m^3 per minute, that is 0·1 m^3 per second, of water through a pipe of cross-sectional area 0·004 m^2 requires the water to have an (average) speed of 0·1/0·004 ms^{-1} = 25 ms^{-1}. Thus in each minute the pump supplies kinetic energy to the amount

$$\tfrac{1}{2} \times 6 \times 10^3 \times (25)^2 \text{ J} = 1{\cdot}875 \times 10^6 \text{ J}.$$

Therefore the total work done each minute is

$$(1{\cdot}1772 + 1{\cdot}875) \times 10^6 \text{ J} = 3{\cdot}0522 \times 10^6 \text{ J}$$

$$\Rightarrow \quad \text{power} = (3{\cdot}0522 \times 10^6/60) \text{ W} = 50{\cdot}87 \text{ kW}.$$

(Note that the calculation assumes the speed of water to be uniform over the cross-section, whereas in fact that would not be so. It can be shown that the actual kinetic energy is greater than that calculated above; hence the real power is greater than 50·87 kW.)

Example 7 The engine of a car develops constant power when it is in motion, and the car experiences a resistance proportional to the square of its velocity. The maximum speed of the car on a level road is U, while the maximum speed up a certain hill of constant slope is $\tfrac{1}{2}U$. Show that the maximum speed down the same hill is λU, where

$$\lambda^3 - \tfrac{7}{4}\lambda - 1 = 0.$$

The various situations are illustrated in Fig. 1.4. Let the constant power developed by the car be P. When it has speed v, the driving force exerted by the engine of the car is P/v. Let M be the mass of the car. On level ground, it follows from (1.1) that

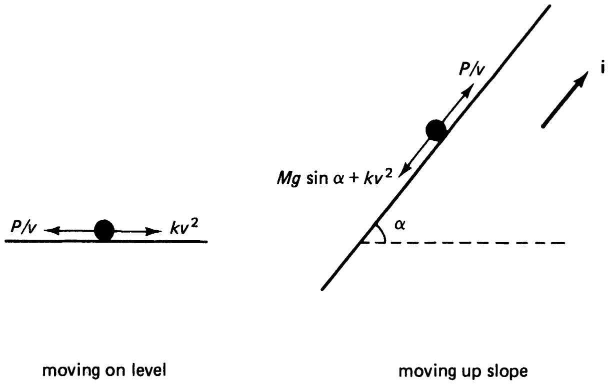


Fig. 1.4

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{P}{v} - kv^2, \quad \text{where } k \text{ is a constant.}$$

When the car is at its maximum speed $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0$. Hence

$$P/U = kU^2 \quad \Rightarrow \quad P = kU^3. \tag{1.33}$$

Suppose the hill has slope α. When the car is moving up the slope with speed v its equation of motion is

$$M\dot{v} = (P/v) - kv^2 - Mg\sin\alpha,$$

since the car's weight has a component $Mg\sin\alpha$ down the slope.
When $\dot{v} = 0$, $v = \frac{1}{2}U$

$$\Rightarrow \quad (2P/U) - \tfrac{1}{4}kU^2 - Mg\sin\alpha = 0. \tag{1.34}$$

Similarly, when the car is moving down the slope,

$$M\dot{v} = (P/v) - kv^2 + Mg\sin\alpha.$$

When $\dot{v} = 0$, $v = \lambda U$

$$\Rightarrow \quad (P/\lambda U) - \lambda^2 kU^2 + Mg\sin\alpha = 0. \tag{1.35}$$

Adding (1.34) and (1.35)

$$\Rightarrow \quad \frac{2P}{U} - \frac{kU^2}{4} + \frac{P}{\lambda U} - \lambda^2 kU^2 = 0.$$

Using $P = kU^3$ from (1.33)

$$\Rightarrow \quad 2kU^2 - \frac{kU^2}{4} + \frac{kU^2}{\lambda} - \lambda^2 kU^2 = 0$$

$$\Rightarrow \quad \lambda^3 - \frac{7}{4}\lambda - 1 = 0.$$

(The solution of this equation is $\lambda \approx 1{\cdot}55$ to 2 decimal places.)

1.7 Some further comments on energy, work and power

In this chapter we have emphasized motion in a straight line. However, as indicated in §1.4 and §1.5, the concepts of energy, work and power are important in other types of motion. Quite generally, consider a particle of mass m moving with velocity $\mathbf{v}$, where $\mathbf{v} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$. We define the *kinetic energy* T of the particle by

$$T = \tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(u_1{}^2 + u_2{}^2 + u_3{}^2). \tag{1.36}$$

Then

$$\dot{T} = \frac{\mathrm{d}T}{\mathrm{d}t} = m(u_1\dot{u}_1 + u_2\dot{u}_2 + u_3\dot{u}_3) = m\dot{\mathbf{v}}\,.\,\mathbf{v}.$$

Hence, using Newton's Second Law, if $\mathbf{F}$ is the force acting on the particle,

$$\dot{T} = \mathbf{F}\,.\,\mathbf{v}. \tag{1.37}$$

We define $\mathbf{F}\,.\,\mathbf{v}$ as the *power*.

Note first that our definitions in this section are consistent with those adopted in the earlier part of this chapter for the special case of motion in a straight line when $\mathbf{v} = v\mathbf{i}$ and $\mathbf{F} = F\mathbf{i}$.

Next, suppose that in the small time interval from t to $t + \delta t$, the particle moves from the point with position vector $\mathbf{r}$ to that with position vector $\mathbf{r} + \boldsymbol{\delta}\mathbf{r}$. The situation is illustrated in Fig. 1.5. By definition, $\mathbf{v} = \lim_{\delta t \to 0} [\boldsymbol{\delta}\mathbf{r}/\delta t]$. Hence $\mathbf{v} \approx \boldsymbol{\delta}\mathbf{r}/\delta t$ and

$$\mathbf{F}\,.\,\mathbf{v} \approx \frac{\mathbf{F}\,.\,\boldsymbol{\delta}\mathbf{r}}{\delta t}. \tag{1.38}$$

We define $\mathbf{F}\,.\,\boldsymbol{\delta}\mathbf{r}$ as the *work done* by the force $\mathbf{F}$ during the time interval δt, and (1.38) shows that power is the rate of doing work, as in our earlier definition. It follows that $\mathbf{F}$ does no work if $\mathbf{F}\,.\,\boldsymbol{\delta}\mathbf{r} = 0$, that is if $\mathbf{F}$ and $\boldsymbol{\delta}\mathbf{r}$ are perpendicular. We met one example of this in §1.4 where the tension in the string $-T\hat{\mathbf{r}}$ was perpendicular to $\boldsymbol{\delta}\mathbf{r} \approx l\delta\theta\hat{\boldsymbol{\theta}}$. We noted then that, consistent with the present interpretation, the tension in the string had no effect on the energy equation (1.23).

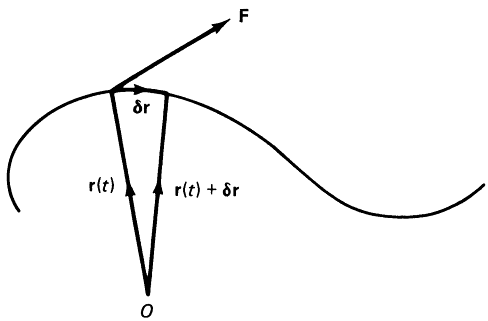


Fig. 1.5

It is possible, in certain circumstances, to define potential energy for the general case considered in this section and then to obtain the principle of conservation of energy. However the subject is too difficult to deal with here. We conclude by stressing that this principle has been derived only for certain motions. In particular, we have not considered cases where $\mathbf{F}$ depends on $\mathbf{v}$, as with air resistance, and the reader is advised that energy conservation does not (generally) then apply.

Exercise 1

1 A pump raises 100 kg of water per second from a depth of 30 m. The water is delivered at a speed of 30 ms^{-1}. Find (a) the potential energy, and (b) the kinetic energy, gained by the water delivered each second; take g as 9·8 ms^{-2}. Neglecting frictional losses, calculate the rate at which the pump is working.

2 A train is travelling along a straight level track at a speed of 36 km h^{-1}. Given that the power exerted by the engine is 500 kW, find the tractive force exerted by the engine. If the train has mass 6×10^5 kg and has an acceleration of 0·05 ms^{-2}, calculate the total resistance to the motion.

3 A load of 2000 kg is being pulled up a slope inclined at an angle α to the horizontal, where $\sin \alpha = 0{\cdot}01$. The coefficient of friction between the load and the slope is 0·04. The acceleration of the load is directed up the slope and has magnitude 0·21 ms^{-2}. Find the tension in the rope, taking $\cos \alpha$ to be 1 and g to be 9·8 ms^{-2}.

The load starts from rest. Find the work done by the rope on the load during the first 20 s of motion, and the power being developed at the end of the first 20 s of motion. After 20 s of motion the rope breaks. How much further does the load travel up the slope?

4 A car moves along a level road. Its speed v is related to its displacement x by the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{P}{v} - kv^2,$$

where P and k are positive constants. What are the likely physical causes of each of the terms on the right-hand side of this equation? What is the maximum possible speed v_0 of the car?

By differentiating $3kx + \ln(P - kv^3)$ with respect to x, verify that the equation is satisfied if $3kx + \ln(P - kv^3) = C$, where C is a constant. Hence show that the distance travelled while accelerating from speed $\frac{1}{3}v_0$ to speed $\frac{2}{3}v_0$ is about $0{\cdot}105k^{-1}$.

5 An elastic string of natural length 0·5 m obeys Hooke's law. The work done in stretching the string from a length 0·75 m to a length 1·25 m is 4 J. Find the energy stored in the string when its length is 1 m.

6 A particle of mass m is attached to one end of an elastic string of natural length a, the other end of which is attached to a fixed point O. When the particle hangs in equilibrium, the total length of the string is $5a/4$. The particle is held at O and then released from rest. Find the speed of the particle when it is a distance $5a/4$ below O, and the maximum extension of the string.

7 A child carrying a basket containing eggs swings his arm in a vertical plane so that the bottom of the basket describes a circle of radius 0·8 m at constant angular speed in a time of 1·5 s. Taking g as 9·8 ms^{-2}, determine whether the eggs fall out of the basket.

8 The length of a simple pendulum is 0·75 m, and the speed at the lowest point is 0·3 ms^{-1}. Taking g as 9·81 ms^{-2}, estimate (a) the period of the oscillation to two decimal places, and (b) the maximum angle between the string and the downward vertical to the nearest 0·1°.

9 The force of attraction of the Earth on a particle of mass m distant x from the centre of the Earth is mga^2/x^2, where a is the radius of the Earth. Prove that the work done when the particle moves from the Earth's surface to a distance x from its centre is

$$mga^2(x^{-1} - a^{-1}).$$

Use the principle of conservation of energy to find an expression for the speed of the particle at distance x from the centre of the Earth, given that it is projected vertically upwards from the Earth's surface with speed $\sqrt{(2ga)}$. Hence prove that the time taken to reach a height h above the Earth's surface is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1 + \frac{h}{a}\right)^{3/2} - 1\right].$$

10 Calculate the work done by the constant force $\mathbf{F} = F(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ when the particle on which it is acting moves from the origin to the point with position $a(-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$.

2 Momentum, impact and impulse

2.1 Introduction

The work in this chapter is again based on Newton's laws, which are first used to establish the principle of conservation of momentum for systems of particles. We also study how to predict the consequences of a sharp blow occurring, for example, when two particles collide.

2.2 Conservation of momentum

The *momentum* $\mathbf{p}$, sometimes called *linear momentum*, of a particle of mass m moving with velocity $\mathbf{v}$ is defined by

$$\mathbf{p} = m\mathbf{v}. \tag{2.1}$$

According to Newton's Second Law $\dot{\mathbf{p}} = \mathbf{F}$. Hence the momentum of a particle moving under no forces is constant, that is the momentum of the particle is conserved. This result may also be deduced from Newton's First Law and the additional assumption, to be made throughout this chapter, that the mass m is also constant.

More generally, consider two particles of mass m_1 and m_2 moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Suppose, further, that the only force acting on the particle of mass m_1 is due to the presence of mass m_2, and vice versa. So, by Newton's Third Law, if $\mathbf{F}$ is the force acting on the particle of mass m_1, then $-\mathbf{F}$ is the force acting on the particle of mass m_2. By Newton's Second Law, applied to each particle separately

$$\dot{\mathbf{p}}_1 = m_1\dot{\mathbf{v}}_1 = \mathbf{F}; \quad \dot{\mathbf{p}}_2 = m_2\dot{\mathbf{v}}_2 = -\mathbf{F}. \tag{2.2}$$

Adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{0}. \tag{2.3}$$

Define the total momentum $\mathbf{P}$ of the system of two particles by

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2. \tag{2.4}$$

By equation (2.3), $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{0}$. Hence

$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \text{ is constant.} \tag{2.5}$$

This is the *principle of conservation of momentum*, derived here for the case of only two particles. However it is easy to extend the proof to show that (2.5) holds for a system of any number of particles provided that the only forces acting are those between different particles of the system.

A restricted version of the principle is often useful. Suppose the two particles are moving in the same straight line with velocities $v_1\mathbf{i}$ and $v_2\mathbf{i}$ respectively, so that the total momentum $\mathbf{P}$, from (2.4), is

$$\mathbf{P} = P_1\mathbf{i} = (m_1v_1 + m_2v_2)\mathbf{i}. \tag{2.6}$$

Provided the components of the forces on each particle in the direction of $\mathbf{i}$ are due only to their mutual interaction, it follows, by considering the components of (2.2), (2.3) and (2.4) in the direction of $\mathbf{i}$, that

$$P_1 = m_1v_1 + m_2v_2 \text{ is constant.} \tag{2.7}$$

A typical illustration of (2.7) is given in the next example.

Example 1 A bullet of mass 0·02 kg is moving horizontally with a speed of 505 ms^{-1} when it hits a stationary wooden block of mass 2 kg which is free to move on a smooth horizontal table. The bullet becomes embedded in the block. Find

(i) the final speed of the block containing the embedded bullet;
(ii) the total loss of kinetic energy.

(i) When the bullet hits the block it penetrates it until it comes to rest relative to it. Let the final speed of the block and bullet be V ms^{-1}. While the bullet is penetrating the block, the force $\mathbf{F}$ exerted on the bullet is unknown and very complicated. However, by Newton's Third Law, the bullet exerts a force $-\mathbf{F}$ on the block. Also, since the table is smooth, no external force is exerted on the block in the direction of motion; in particular there is no frictional force between the block and the table. Hence we can apply (2.7) in the form:

$$\text{original value of } P_1 = \text{final value of } P_1$$

$$\Rightarrow \quad (0{\cdot}02 \times 505) + (2 \times 0) = (0{\cdot}02 + 2)\,V \quad \Rightarrow \quad V = 5\ \mathrm{ms}^{-1}.$$

(ii) Original kinetic energy $= (\frac{1}{2} \times 0{\cdot}02 \times 505^2)\ \mathrm{J} = 2550{\cdot}25\ \mathrm{J}$

$$\text{final kinetic energy} = (\tfrac{1}{2} \times 2{\cdot}02 \times 5^2)\ \mathrm{J} = 25{\cdot}25\ \mathrm{J}$$

$$\Rightarrow \quad \text{loss of kinetic energy} = 2525\ \mathrm{J}.$$

We note from the result of part (ii) of this example that *kinetic energy is not conserved* in this motion. This is because other forms of energy, especially heat and sound, are generated while the bullet penetrates the block, and we are not yet in a position to take account of these.

Another interpretation of (2.5) is important. Let $M = m_1 + m_2$ be the total mass of the system of two particles, and let their position vectors be $\mathbf{r}_1$ and $\mathbf{r}_2$. Define $\mathbf{R}$ by

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \quad \Rightarrow \quad \mathbf{R} = \frac{(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)}{(m_1 + m_2)}. \tag{2.8}$$

We shall meet (2.8) in the next chapter, where it will be seen that the point with position vector $\mathbf{R}$ is called the *centre of mass* of the system.

Differentiating both sides of (2.8) with respect to t, and using $\dot{\mathbf{r}}_1 = \mathbf{v}_1$ and $\dot{\mathbf{r}}_2 = \mathbf{v}_2$, we find from (2.5) that

$$M\dot{\mathbf{R}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{constant}. \tag{2.9}$$

Hence $\dot{\mathbf{R}}$ is constant, from which we deduce that the centre of mass is moving with constant velocity. Once again it is easy to extend this result to a system of any number of particles provided the only forces acting are those between different particles of the system.

Example 2 Two particles of mass m_1 and m_2 have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. State the physical significance of the three quantities T, $\mathbf{V}$, $\mathbf{W}$ defined by:

$$T = \tfrac{1}{2}(m_1\mathbf{v}_1{}^2 + m_2\mathbf{v}_2{}^2); \quad \mathbf{V} = \frac{(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)}{(m_1 + m_2)}; \quad \mathbf{W} = \mathbf{v}_1 - \mathbf{v}_2. \tag{2.10}$$

Show that

$$T = \frac{1}{2}\left[(m_1 + m_2)\mathbf{V}^2 + \left(\frac{m_1 m_2}{m_1 + m_2}\right)\mathbf{W}^2\right]. \tag{2.11}$$

A bomb of mass $(m_1 + m_2)$ explodes into two bodies of masses m_1 and m_2. The total kinetic energy of the two bodies after the explosion is E greater than the kinetic energy of the original bomb. Find the magnitude of the relative velocity of the two bodies in terms of E, m_1, m_2.

$$(1.36) \quad \Rightarrow \quad T \text{ is the total kinetic energy;}$$

$$(2.8) \quad \Rightarrow \quad \mathbf{V} \text{ is the velocity of the centre of mass.}$$

$\mathbf{W}$ is the velocity of the mass m_1 relative to that of the mass m_2.

$$(2.10) \quad \Rightarrow \quad \mathbf{V}(m_1 + m_2) = (m_1 + m_2)\mathbf{v}_2 + m_1\mathbf{W}$$

$$\Rightarrow \quad \mathbf{v}_2 = \mathbf{V} - \frac{m_1}{(m_1 + m_2)}\mathbf{W}.$$

Similarly

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2}{(m_1 + m_2)}\mathbf{W}.$$

Therefore

$$\mathbf{v}_1{}^2 = \mathbf{V}^2 + \frac{2m_2}{(m_1 + m_2)}\mathbf{V}.\mathbf{W} + \frac{m_2{}^2}{(m_1 + m_2)^2}\mathbf{W}^2;$$

$$\mathbf{v}_2{}^2 = \mathbf{V}^2 - \frac{2m_1}{(m_1 + m_2)}\mathbf{V}.\mathbf{W} + \frac{m_1{}^2}{(m_1 + m_2)^2}\mathbf{W}^2.$$

If we substitute these expressions into the equation for T in (2.10), we obtain

$$T = \frac{1}{2}\left[m_1\mathbf{V}^2 + \frac{2m_1m_2}{(m_1+m_2)}\mathbf{V}.\mathbf{W} + \frac{m_1{m_2}^2}{(m_1+m_2)^2}\mathbf{W}^2 + m_2\mathbf{V}^2 - \frac{2m_1m_2}{(m_1+m_2)}\mathbf{V}.\mathbf{W} + \frac{m_2{m_1}^2}{(m_1+m_2)^2}\mathbf{W}^2\right]$$

$$= \frac{1}{2}\left[(m_1+m_2)\mathbf{V}^2 + \left(\frac{m_1m_2}{m_1+m_2}\right)\mathbf{W}^2\right] \text{ as required.}$$

If, during the explosion, there is a force $\mathbf{F}$ exerted on m_1, then by Newton's Third Law there is a force $-\mathbf{F}$ exerted on m_2. Hence momentum is conserved. By (2.9) this means that the value of $\mathbf{V}$ before the explosion is equal to the value of $\mathbf{V}$ after the explosion. Also, before the explosion $\mathbf{W} = \mathbf{0}$. Hence, from (2.11), the increase E in kinetic energy satisfies

$$E = \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)\mathbf{W}^2 \quad \Rightarrow \quad |\mathbf{W}| = \sqrt{\left(\frac{2E(m_1+m_2)}{m_1m_2}\right)},$$

which is the required answer, since $|\mathbf{W}|$ is the magnitude of the relative velocity. (Once again kinetic energy is not conserved; in this case it increases by an amount E which has been converted from the chemical energy released by the explosive.)

2.3 Collisions

Consider two spheres of the same size but of different masses m_1 and m_2. Let them be moving along the line joining their centres, parallel to $\mathbf{i}$, with velocities $v_1\mathbf{i}$ and $v_2\mathbf{i}$. Here, one or both of v_1 and v_2 may be negative. In some cases, two of which are illustrated in Fig. 2.1, *collision* will occur. As a result of the collision, the two spheres may stick together (in which case their final speed can be obtained as in Example 1), or they may separate. It is with the latter case that this section is concerned. Let the velocities of the spheres of masses m_1 and m_2 after the collision be $w_1\mathbf{i}$ and $w_2\mathbf{i}$ respectively, as shown in Fig. 2.2. We wish to find w_1, w_2 in terms of v_1, v_2, m_1, m_2.

Momentum is conserved, according to the argument used earlier in this section.

$$(2.7) \quad \Rightarrow \quad m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2. \tag{2.12}$$

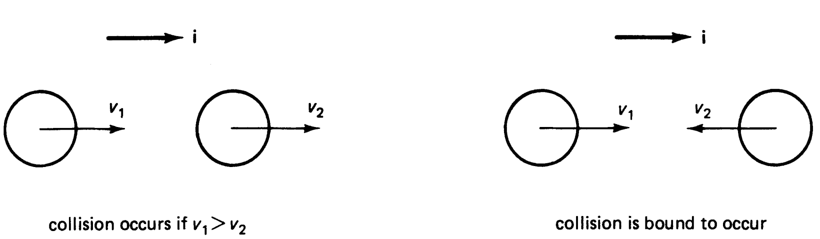


Fig. 2.1

Fig. 2.2

But (2.12) only provides *one* equation for the *two* unknowns w_1 and w_2. One more equation is needed. Experiments show that for many spheres there is a simple relationship between the speed of approach, namely $v_1 - v_2$, and the speed of separation, namely $w_2 - w_1$. This relationship is that

The speed of separation is equal to a constant e times the speed of approach, or, in symbols,

$$(w_2 - w_1) = e(v_1 - v_2), \tag{2.13}$$

where e is a positive constant known as the *coefficient of restitution*. The law expressed by (2.13) is often called *Newton's law of impact*. The value of e is dependent on the materials of which the particles are made, but always satisfies the inequality

$$0 \leqslant e \leqslant 1. \tag{2.14}$$

When $e = 1$ the particles are said to be *perfectly elastic* since the relative speed has the same magnitude before and after collision; when $e = 0$ the particles are said to be *inelastic* (sometimes *perfectly inelastic*) since then, according to (2.13), $w_1 = w_2$ and the particles have the same velocity after collision, that is they stick together (or coalesce).

Equations (2.12) and (2.13) are a pair of simultaneous equations from which w_1 and w_2 can be determined, as illustrated in the next example.

Example 3 A sphere A of mass m, moving with speed $7V$, collides with a sphere B of the same radius and mass $2m$, moving in the same direction with speed V. Given that the coefficient of restitution is $\frac{3}{4}$, find the speeds of A and B after the collision. Find also the loss in kinetic energy.

Let w_1 and w_2 be the speeds of A and B respectively after the collision. In (2.12) and (2.13) we have $m_1 = m$, $m_2 = 2m$, $v_1 = 7V$, $v_2 = V$, $e = \frac{3}{4}$.

$$(2.12) \quad \Rightarrow \quad mw_1 + 2mw_2 = 7mV + 2mV \quad \Rightarrow \quad w_1 + 2w_2 = 9V.$$

$$(2.13) \quad \Rightarrow \quad w_1 - w_2 = -(\tfrac{3}{4})(7V - V) \quad \Rightarrow \quad w_1 - w_2 = -(9V/2)$$

$$\Rightarrow \quad 3w_2 = 27V/2 \quad \Rightarrow \quad w_2 = (9V/2), \quad w_1 = 0.$$

Kinetic energy before collision $= \frac{1}{2}m_1v_1{}^2 + \frac{1}{2}m_2v_2{}^2 = (51/2)mV^2$.

Kinetic energy after collision $= \frac{1}{2}m_1 w_1{}^2 + \frac{1}{2}m_2 w_2{}^2 = (81/4)mV^2$

$\Rightarrow$ loss of kinetic energy $= (21/4)mV^2$.

It is important to understand that Newton's law of impact, equation (2.13), relates the situation *before* collision to that *after* collision. It does not tell us anything about what happens *during* collision. In fact careful observations show that the colliding spheres are deformed during collision, eventually regaining their shapes. Theoretical study of such deformations is very difficult, certainly beyond the scope of this book. However heat and sound energy are generated during the deformation and, as indicated following Example 1, p. 19, these are supplied by the kinetic energy of the particles. Thus kinetic energy must be lost as a result of the collision.

The amount of kinetic energy lost can be calculated using (2.11). Since, according to (2.9), the velocity of the centre of mass V is conserved during the collision, the loss ΔT of kinetic energy is given by

$$\Delta T = \frac{1}{2}\frac{m_1 m_2}{(m_1 + m_2)}\left[\mathbf{W}_b{}^2 - \mathbf{W}_a{}^2\right],$$

where $\mathbf{W}_b$ and $\mathbf{W}_a$ are the relative velocities of the two particles before and after collision. When the motion of the particles throughout is along the straight line parallel to $\mathbf{i}$, we have, using the notation of this section,

$$\mathbf{W}_b = (v_1 - v_2)\mathbf{i}, \quad \mathbf{W}_a = (w_1 - w_2)\mathbf{i}.$$

So, from equation (2.13),

$$\mathbf{W}_b{}^2 - \mathbf{W}_a{}^2 = (v_1 - v_2)^2 - (w_1 - w_2)^2 = (1 - e^2)(v_1 - v_2)^2.$$

Substitution into the above equation gives

$$\Delta T = \frac{1}{2}\frac{m_1 m_2}{(m_1 + m_2)}(1 - e^2)(v_1 - v_2)^2. \tag{2.15}$$

Since $0 \leqslant e \leqslant 1$ from (2.14), we see from (2.15) that ΔT is positive unless $e = 1$, that is unless the particles are perfectly elastic. When $m_1 = m$, $m_2 = 2m$, $v_1 = 7V$, $v_2 = V$, $e = \frac{3}{4}$, the values in Example 3, equation (2.15) gives

$$\Delta T = \frac{1}{2} \times \frac{2m^2}{3m} \times \left(1 - \frac{9}{16}\right) \times 36V^2 = \frac{21}{4}mV^2,$$

in agreement with the final result of Example 3.

Newton's law of impact is also observed to apply when a particle drops vertically and collides with a stationary flat surface. In such a case, however, m_2 is the mass of the Earth and is therefore much greater than m_1, the mass of the particle. We can rearrange (2.12) in the form

$$(w_2 - v_2) = \frac{m_1}{m_2}(v_1 - w_1),$$

and when m_1/m_2 is extremely small we see that w_2 is approximately equal to v_2. Since the surface is stationary, $v_2 = 0$, we can take

$$w_2 = v_2 = 0$$

as a very accurate approximation. Equation (2.13) then reduces to

$$w_1 = -ev_1. \tag{2.16}$$

The minus sign in (2.16) is required because the velocity of the particle after collision is directed in the opposite direction (upwards) to that before collision (downwards).

Example 4 A particle of mass m is dropped from a height h above a massive horizontal table. The coefficient of restitution between the particle and the table is e, with $0 \leqslant e \leqslant 1$. Find the total distance travelled by the particle before it finally comes to rest on the table. Verify that the total kinetic energy lost at collisions is mgh, and explain why this result could have been anticipated.

Let the magnitudes of the speeds of the particle just before and just after the nth collision with the table be V_n and W_n respectively.

$$\left.\begin{array}{rcl} (2.16) & \Rightarrow & W_n = eV_n. \\ \text{conservation of energy} & \Rightarrow & V_n = W_{n-1}. \end{array}\right. \tag{2.17}$$

It follows by repeated use of (2.17) that

$$W_n = eV_n = eW_{n-1} = e^2 V_{n-1} = \ldots = e^n V_1. \tag{2.18}$$

Let h_n be the maximum height above the table reached by the particle between the nth and $(n + 1)$th collision. By (1.6),

$$h_n = W_n^2/(2g).$$

Between the nth and $(n + 1)$th collisions the particle travels a distance $2h_n = W_n^2/g$. Before the first collision it falls a distance h. Let d be the total distance travelled. Then

$$d = h + 2h_1 + 2h_2 + \ldots = h + (W_1^2 + W_2^2 + \ldots)/g.$$

$$(2.18) \quad \Rightarrow \quad d = h + (V_1^2/g)(e^2 + e^4 + \ldots).$$

But $(e^2 + e^4 + \ldots)$ is the sum of an infinite geometric series of common ratio e^2 and first term e^2. Hence $(e^2 + e^4 + \ldots) = e^2/(1 - e^2)$. By (1.6), $V_1^2 = 2gh$;

$$\Rightarrow \quad d = h + 2h\left(\frac{e^2}{1 - e^2}\right) = h\left(\frac{1 + e^2}{1 - e^2}\right).$$

The kinetic energy lost at the nth collision is

$$\Delta T_n = \tfrac{1}{2}m(V_n^2 - W_n^2) = \tfrac{1}{2}m(1 - e^2)V_n^2 = \tfrac{1}{2}m(1 - e^2)(e^{n-1}V_1)^2$$

by (2.18). Hence the total loss is

$$\Delta T = \Delta T_1 + \Delta T_2 + \ldots = \tfrac{1}{2}mV_1^2(1-e^2)(1+e^2+\ldots) = \tfrac{1}{2}mV_1^2 = mgh,$$

as required. We can anticipate this result for the following reason. Initially the particle has potential energy mgh and zero kinetic energy. Finally the particle is at rest on the table so both its potential and kinetic energies are zero. Hence its total loss of energy is mgh. This has been converted into other forms of energy, including heat and sound.

2.4 Impulses

Let us reconsider the situation described in Example 4 and let $\mathbf{i}$ denote a unit vector vertically downwards. Just before the first collision the particle has velocity $V_1\mathbf{i}$, and just after it has velocity $-W_1\mathbf{i} = -eV_1\mathbf{i}$, using (2.16). Hence the momentum $\mathbf{p}$, defined in (2.1), has changed from $mV_1\mathbf{i}$ to $-emV_1\mathbf{i}$, a decrease of $mV_1(1+e)\mathbf{i}$. This decrease has occurred entirely as a result of the force exerted by the table on the particle during the collision process. As explained following Example 3, we cannot however expect to be able to calculate this force throughout the time interval when the particle is in contact with the table.

Therefore we are only able to describe the effect of the table on the particle, and we say that the table has given the particle an *impulse* $\mathbf{I}$. The impulse is defined by the equation

$$\mathbf{I} = \mathbf{p}_a - \mathbf{p}_b, \tag{2.19}$$

where $\mathbf{p}_a$ and $\mathbf{p}_b$ are the momenta of the particle after and before the impact respectively. In Example 4, $\mathbf{I} = -emV_1\mathbf{i} - mV_1\mathbf{i} = -mV_1(1+e)\mathbf{i}$.

We can look at the concept of an impulse from a different point of view. By Newton's Second Law, the momentum $\mathbf{p} = m\mathbf{v}$ of a particle satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \mathbf{F}. \tag{2.20}$$

Now suppose that $\mathbf{F}$ is the force occurring as the result of an impact beginning at time t_1 and ending at time t_2. Then, using the notation introduced in (2.19), the left-hand side of (2.20) is $\mathbf{p}_a - \mathbf{p}_b$. Hence the impulse $\mathbf{I}$ satisfies

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t, \tag{2.21}$$

and so the impulse $\mathbf{I}$ is the integral of the force $\mathbf{F}$ with respect to time.

Normally the duration of an impact is very short so that $t_2 - t_1 = \delta t$ is small. Hence the right-hand side of (2.21) is also small, unless $\mathbf{F}$ has very large magnitude. Forces such as gravity and air resistance do not have large magnitudes, but the forces directly associated with impact and collision do. Therefore the value of the change in momentum due to an impact or collision depends only on these impact forces, and not on forces such as gravity or air resistance.

There are many common situations where the concept of an impulse is relevant, including those in sport involving the striking of a ball. For example, a skilful tennis player learns by experience how to impart different impulses to a ball with his racket, so that he can vary the speed, direction and spin of the ball.

Example 5 A particle of mass 2 kg has a velocity $(3\mathbf{i} + 4\mathbf{j})$ ms^{-1} when it receives a blow, after which its velocity is $(4\mathbf{j} + 4\mathbf{k})$ ms^{-1}. Calculate the magnitude of the impulse given to the particle.

Let $\mathbf{p}_b$ and $\mathbf{p}_a$ be the momenta of the particle before and after the blow respectively.

$$\mathbf{p}_b = 2(3\mathbf{i} + 4\mathbf{j})\ \text{kgms}^{-1}; \quad \mathbf{p}_a = 2(4\mathbf{j} + 4\mathbf{k})\ \text{kgms}^{-1}.$$

$$(2.19) \quad \Rightarrow \quad \mathbf{I} = (-6\mathbf{i} + 8\mathbf{k})\ \text{kgms}^{-1}.$$

$$|\mathbf{I}| = \sqrt{(6^2 + 8^2)}\ \text{kgms}^{-1} = 10\ \text{kgms}^{-1}.$$

Example 6 Two particles A and B of masses m and M respectively are connected by a light, rigid rod. They lie on a smooth, horizontal table and the particle B is given a horizontal impulse $\mathbf{I}$ which sets both particles in motion, A with velocity $\mathbf{w}$, and B with velocity $\mathbf{W}$. Explain why

$$\mathbf{w} = (\mathbf{W}\,.\,\mathbf{n})\mathbf{n}, \tag{2.22}$$

where $\mathbf{n}$ is a unit vector in the direction of motion of A. Show that

$$\mathbf{w} = \frac{(\mathbf{I}\,.\,\mathbf{n})\mathbf{n}}{(m + M)},$$

and obtain the corresponding result for $\mathbf{W}$.

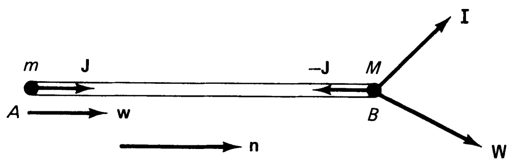


Fig. 2.3

The situation is illustrated in Fig. 2.3 (where the plane of the figure is the plane of the table). As a result of the impulse $\mathbf{I}$, tensions (or thrusts) are set up in the rod which act for just so long as $\mathbf{I}$ acts. In other words, an *impulsive tension* (or *thrust*) $\mathbf{J}$ is set up in the rod in response to $\mathbf{I}$. Applying (2.19) to each of A and B we obtain

$$\mathbf{J} = m\mathbf{w}, \quad \mathbf{I} - \mathbf{J} = M\mathbf{W}. \tag{2.23}$$

Since the rod is rigid, its length cannot change. The velocity $\mathbf{w}$ of A is parallel to $\mathbf{n}$; thus $\mathbf{w} = |\mathbf{w}|\mathbf{n}$. Hence $|\mathbf{w}|$ must be equal to the component of $\mathbf{W}$ along $\mathbf{n}$, that is $\mathbf{W}.\mathbf{n}$. Therefore

$$|\mathbf{w}| = \mathbf{W}.\mathbf{n} \quad \Rightarrow \quad \mathbf{w} = (\mathbf{W}.\mathbf{n})\mathbf{n},$$

which is (2.22).

Since $\mathbf{J}$ is parallel to $\mathbf{n}$, it follows that $(\mathbf{J}.\mathbf{n})\mathbf{n} = \mathbf{J}$.

$$(2.23) \quad \Rightarrow \quad (\mathbf{I}.\mathbf{n})\mathbf{n} - \mathbf{J} = M(\mathbf{W}.\mathbf{n})\mathbf{n} = M\mathbf{w}$$

$$\Rightarrow \quad (\mathbf{I}.\mathbf{n})\mathbf{n} - m\mathbf{w} = M\mathbf{w}.$$

This reduces to the required expression for $\mathbf{w}$.

$$(2.23) \quad \Rightarrow \quad \mathbf{J} = \left(\frac{m}{m+M}\right)(\mathbf{I}.\mathbf{n})\mathbf{n}.$$

$$(2.23) \quad \Rightarrow \quad \mathbf{W} = \frac{1}{M}\left[\mathbf{I} - \left(\frac{m}{m+M}\right)(\mathbf{I}.\mathbf{n})\mathbf{n}\right].$$

Exercise 2

1 A particle A has mass 2 kg and velocity $(4\mathbf{i} + 3\mathbf{j})\ \mathrm{ms}^{-1}$. Calculate the magnitude of the momentum of A. A second particle B has mass 1 kg and velocity $(-\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\ \mathrm{ms}^{-1}$. The particles collide and stick together. Calculate the magnitude of the momentum of the resulting combined particle.

2 A bullet of mass 0·05 kg moving horizontally with speed 321 ms^{-1} strikes a stationary block of mass 16 kg which is free to slide without rotation on a smooth, horizontal table. The bullet remains in the block. Calculate the total loss of kinetic energy.

3 A shell of mass m is fired horizontally by a gun of mass km which is free to recoil on horizontal ground. The total kinetic energy imparted to the shell and the gun is E. Calculate the speed of the shell when it leaves the gun.

4 A small, smooth sphere of mass m and moving with speed V on a smooth, horizontal table collides directly with a second smooth sphere of the same radius as the first but of mass nm and moving with speed kV along the same straight line in the same direction. The first sphere is brought to rest by the impact. Show that the coefficient of restitution is $(1 + kn)/(n - kn)$.

5 A golf ball, initially at rest, is dropped on to a horizontal surface and bounces directly up again with speed v. Given that the coefficient of restitution between the ball and the surface is e, show that the ball will go on bouncing for a time

$$\frac{2v}{g}\frac{1}{(1-e)}$$

after the first impact.

6 Three small, smooth spheres A, B, C of equal radii, and of masses m, $2m$, $4m$ respectively, lie at rest and separated from one another on a smooth, horizontal table in the order A, B, C with their centres in a straight line. The coefficient of restitution between any two spheres is e. Sphere A is projected with speed U directly towards sphere B. Find the speed of each sphere just after C is set in motion.

Given that $e \geqslant \frac{1}{2}$, determine the number of collisions that take place.

7 A small, smooth ball moves on a smooth, horizontal table and strikes an identical sphere lying at rest on the table at a distance d from a vertical wall, the impact being along the line of centres and perpendicular to the wall. Prove that the next impact between the spheres will take place at a distance

$$2de^2/(1 + e^2)$$

from the wall, where e is the coefficient of restitution for all impacts.

8 A shell of mass $4M$ is projected from the origin with initial velocity $V(6\mathbf{i} + 4\mathbf{j})$, where $\mathbf{i}$ and $\mathbf{j}$ are respectively horizontal and vertically upward unit vectors. There is no air resistance. At the instant when the shell is at its greatest height, it explodes into two pieces. One piece has mass $3M$ and velocity $V(5\mathbf{i} + \mathbf{j})$ just after the explosion. Find the time that elapses before the two pieces reach the ground (which is the horizontal plane through the origin).

Draw a sketch to show the path of the centre of mass of the shell from the instant of the explosion to the instant at which both pieces are at rest on the ground.

9 A girl, standing on horizontal ground, throws a ball from a height of 1 m towards a smooth, vertical, high wall, the bottom of which is 6 m from the girl's feet. The initial velocity of the ball has a magnitude of 9 ms^{-1} and is inclined at 45° above the horizontal. Show that the ball strikes the wall at a height of about 2·64 m above the ground, and that just before striking the wall its vertical component of velocity is about 2·88 ms^{-1} downwards. Take g as 9·8 ms^{-2}.

Given that the coefficient of restitution between the ball and the wall is $\frac{1}{2}$, find the horizontal distance from the bottom of the wall of the point at which the ball strikes the ground.

10 A small ring A of mass m is free to slide on a fixed, smooth, horizontal wire. A light, inextensible string of length $2a$ has one end attached to A and the other end to a particle B of mass $2m$. Initially B is held close to the wire at a distance $a\sqrt{3}$ from A. If B is allowed to fall freely, find the magnitude of the impulsive tension in the string when it becomes taut.

Solve the same problem when, instead of being free to slide along the wire, A is fixed.

3 Centres of mass

3.1 Introduction

Previously we have considered only the motion of particles, which are defined to be bodies whose size does not influence their motion. In the remaining chapters we shall be concerned with bodies whose size has an important effect on their motion. In this chapter we discover how to determine the *centre of mass* of a body. As we shall see in Chapter 4, the importance of the centre of mass of a body is that it is the point of the body through which its weight is assumed to act. Other terms for centre of mass are *centroid* and *centre of gravity*.

3.2 The centre of mass of a system of particles

We defined the centre of mass of a system of two particles in equation (2.8). Let the two particles have masses m_1, m_2 and position vectors $\mathbf{r}_1, \mathbf{r}_2$ respectively. Throughout the rest of this book we shall use the symbol $\bar{\mathbf{r}}$, rather than the symbol $\mathbf{R}$ of equation (2.8), for the position vector of the centre of mass. For the present case, of a system of two particles, we define $\bar{\mathbf{r}}$ by

$$\bar{\mathbf{r}} = \frac{(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)}{(m_1 + m_2)}. \tag{3.1}$$

As a special case of this definition, suppose $m_1 = m_2$. Then

$$(3.1) \quad \Rightarrow \quad \bar{\mathbf{r}} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2). \tag{3.2}$$

Thus the centre of mass of a system of two particles of equal mass is the mid-point of the line joining them.

We can generalize the definition in (3.1) to a system of n particles of masses $m_1, m_2, \ldots, m_n$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. Then the centre of mass has position vector $\bar{\mathbf{r}}$ defined by

$$\bar{\mathbf{r}} = \frac{(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n)}{(m_1 + m_2 + \ldots + m_n)} = \frac{\Sigma m\mathbf{r}}{\Sigma m}, \tag{3.3}$$

where we have introduced the capital Greek sigma to indicate summation over all n particles. Obviously (3.1) is the special case of (3.3) for which $n = 2$.

It is possible to show that the point whose position vector is given by (3.3) is independent of the choice of the origin O with respect to which position vectors are measured.

Example 1 The vertices of a triangle A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Three equal particles, each of mass m, are placed at the vertices

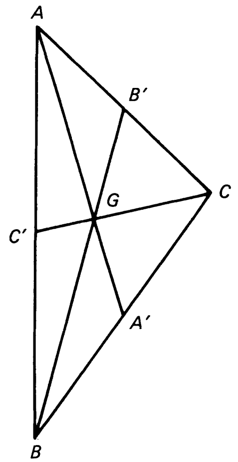


Fig. 3.1

of the triangle. Show that the centre of mass of the system of three particles is the point G where the medians of $\triangle ABC$ meet.

The situation is illustrated in Fig. 3.1. By (3.3) with $m_1 = m_2 = m_3 = m$ and $\mathbf{r}_1 = \mathbf{a}$, $\mathbf{r}_2 = \mathbf{b}$, $\mathbf{r}_3 = \mathbf{c}$, we see that

$$\bar{\mathbf{r}} = \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

We have to show that $\bar{\mathbf{r}} = \mathbf{g}$, where $\mathbf{g}$ is the position vector of G. Let A' be the mid-point of BC; the position vector of A' is $\tfrac{1}{2}(\mathbf{b} + \mathbf{c})$. Then any point on the median AA' has position vector

$$(1 - \lambda)\mathbf{a} + \tfrac{1}{2}\lambda(\mathbf{b} + \mathbf{c})$$

for some value of λ. By choosing $\lambda = \tfrac{2}{3}$ we see that the point with position vector $\tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ lies on AA'. By symmetry this point also lies on the other two medians BB' and CC'. Hence it is G. Therefore, as required,

$$\mathbf{g} = \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \bar{\mathbf{r}}.$$

3.3 The centre of mass of a continuous distribution of mass

We begin with an example. Figure 3.2 shows a sketch of a circular disc of radius a which has been cut from a large piece of metal of uniform small thickness. We want to determine the centre of mass of this disc. By symmetry we would expect this to be at O, the centre of the disc. Let us show how we can obtain this result from the above definition. We can regard the disc as composed of a large number of thin rings. One of these rings, of radius r and thickness δr, is shown on the figure. Now consider two straight lines through O which are

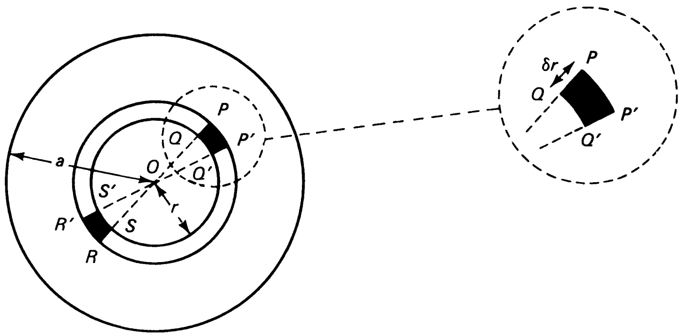


Fig. 3.2

very close together. One of these lines meets the ring in the points P, Q, S, R and the other meets it in the points P', Q', S', R', where

$$OQ = OQ' = OS = OS' = r \quad \text{and} \quad OP = OP' = OR = OR' = r + \delta r.$$

These lines separate two equal small regions of the disc, one with corners P, P', Q', Q and the other with corners R, R', S', S. These two regions are small, so they can be regarded as particles. By equation (3.2) their centre of mass is at the mid-point of the line joining them, that is at O. We can divide the ring up into many pairs of small regions, the centre of mass of each pair being at O. It therefore follows that the centre of mass of the whole ring is at O, and, because the complete disc is composed of a large number of rings, that, as anticipated earlier, the centre of mass of the disc is also at O.

This example shows that we can continue to apply our definition (3.3) for a continuous body, provided we can regard the body as composed of a large number of small elements, each of which can be regarded as a particle. Similar arguments enable us to show, for example, that the centre of mass of a uniform rod is at its mid-point, and that the centre of mass of a uniform sphere is at the centre of the sphere.

The next example shows how we can make use both of (3.3) and symmetry arguments to find the centre of mass of a composite body.

Example 2 A sledge-hammer consists of a uniform cylindrical shaft of length 1 m and mass 0·6 kg, one end of which passes through a heavy metal head of mass 5·4 kg. The centre of mass of the metal head is 0·95 m from the top of the shaft. Find the position of the centre of mass of the sledge-hammer.

The sledge-hammer is sketched in Fig. 3.3, where G_S and G_H are the centres of mass of the shaft and head respectively. We measure position vectors relative to the top of the shaft O and let $\mathbf{i}$ be a unit vector along the axis of the shaft. Then if $\bar{\mathbf{r}}_S$ and $\bar{\mathbf{r}}_H$ are the position vectors of G_S and G_H respectively,

$$\bar{\mathbf{r}}_S = 0{\cdot}5\mathbf{i} \text{ m}, \quad \bar{\mathbf{r}}_H = 0{\cdot}95\mathbf{i} \text{ m}.$$

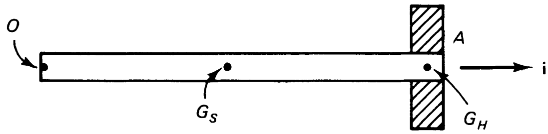


Fig. 3.3

By symmetry, the position vector $\bar{\mathbf{r}}$ of the centre of mass G of the whole sledge-hammer is parallel to $\mathbf{i}$. Suppose $\bar{\mathbf{r}} = \bar{x}\mathbf{i}$. As far as the determination of $\bar{\mathbf{r}}$ is concerned, we can regard the shaft as a mass of 0·6 kg located at G_S, and the head as a mass of 5·4 kg located at G_H.

$$(3.2) \quad \Rightarrow \quad \bar{\mathbf{r}} = \bar{x}\mathbf{i} = \left\{\frac{(0{\cdot}6)(0{\cdot}5) + (5{\cdot}4)(0{\cdot}95)}{0{\cdot}6 + 5{\cdot}4}\right\}\mathbf{i}\ \mathrm{m}$$

$$\Rightarrow \quad \bar{x} = \frac{0{\cdot}3 + 5{\cdot}13}{6}\ \mathrm{m} = 0{\cdot}905\ \mathrm{m}.$$

The basic method used in Example 2 can be applied in a different way as shown in Example 3.

Example 3 Figure 3.4 shows a uniform square lamina $ABCD$ of side $2a$ and mass $4m$ from which a square $EOFC$ of side a has been removed. Find the position of the centre of mass G of the remaining L-shaped lamina. (A *lamina* is another term for a plane sheet of material of uniform small thickness.)

Let position vectors be measured from O, the centre of $ABCD$, and let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors along OE and OF respectively.

We regard $ABCD$ as composed of the L-shaped lamina of mass $3m$ and centre of mass G, together with the square $EOFC$ of mass m and centre of mass G'. By symmetry, both G and G' lie on AO. Let G and G' have position vectors $\bar{\mathbf{r}}, \bar{\mathbf{r}}'$ respectively.

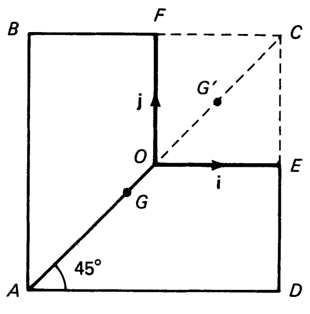


Fig. 3.4

$$OE = a \quad \Rightarrow \quad OC = a\sqrt{2} \quad \Rightarrow \quad OG' = \tfrac{1}{2}a\sqrt{2}$$
$$\Rightarrow \quad \bar{\mathbf{r}}' = \tfrac{1}{2}a(\mathbf{i} + \mathbf{j}).$$

Since the centre of mass of $ABCD$ is at O we have, using (3.1),

$$\mathbf{0} = \frac{3m\bar{\mathbf{r}} + \frac{1}{2}ma(\mathbf{i} + \mathbf{j})}{4m} \quad \Rightarrow \quad \bar{\mathbf{r}} = -\frac{1}{6}a(\mathbf{i} + \mathbf{j}).$$

Hence G lies on AO at a distance $\sqrt{\left\{\left(\frac{5a}{6}\right)^2 + \left(\frac{5a}{6}\right)^2\right\}} = \frac{5\sqrt{2}a}{6}$ from A.

3.4 The use of integration to determine centres of mass of continuous bodies

For many bodies we have to use integration to determine $\bar{\mathbf{r}}$. In all cases the principle is the same and is based on the fundamental definition (3.3). We illustrate the technique by several examples.

Example 4 Find the position of the centre of mass of a uniform lamina in the shape of a semicircle of radius a.

The lamina is illustrated in Fig. 3.5. We take axes Ox, Oy and unit vectors $\mathbf{i}$, $\mathbf{j}$ as shown, so that the curved boundary has equation $y = \sqrt{(a^2 - x^2)}$. By symmetry the centre of mass G lies on Oy so that its position vector $\bar{\mathbf{r}}$ is of the form

$$\bar{\mathbf{r}} = \bar{y}\mathbf{j}.$$

We regard the lamina as made up of a series of strips of small width δx of which one is indicated. If $ON = x$, then $PN = \sqrt{(a^2 - x^2)}$. Hence, if δA is the area of the strip, we see that

$$\delta A \approx \sqrt{(a^2 - x^2)}\delta x.$$

Let the mass of the strip be δm, and suppose the total mass of the lamina is m. Since the area of the semicircle is $\frac{1}{2}\pi a^2$ we have

$$\frac{\delta m}{m} = \frac{\delta A}{\frac{1}{2}\pi a^2} \quad \Rightarrow \quad \delta m \approx \frac{2m}{\pi a^2}\sqrt{(a^2 - x^2)}\delta x.$$

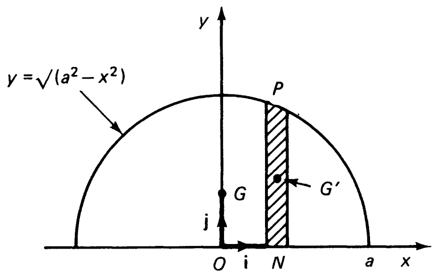


Fig. 3.5

Also the centre of mass G' of the strip has position vector $\bar{\mathbf{r}}' = x\mathbf{i} + \frac{1}{2}\sqrt{(a^2 - x^2)}\mathbf{j}$. Let us suppose that there are n strips making up the lamina. To determine the position of G, we can, following the method introduced in Examples 2 and 3, regard each strip as a mass δm located at G'. Then, using (3.3), we have

$$\bar{\mathbf{r}} = \frac{\Sigma m\bar{\mathbf{r}}'}{\Sigma \delta m} = \frac{\dfrac{2m}{\pi a^2}\Sigma\sqrt{(a^2 - x^2)}\delta x\bar{\mathbf{r}}'}{m},$$

since $\Sigma\delta m = m$. We are interested in the y component of this equation, which, using the above expression for $\bar{\mathbf{r}}'$, is

$$\bar{y} = \frac{1}{\pi a^2}\Sigma(a^2 - x^2)\delta x.$$

When δx is very small, the summation can be replaced by integration in the normal way, so that

$$\bar{y} = \frac{1}{\pi a^2}\int_{-a}^{a}(a^2 - x^2)\,\mathrm{d}x = \frac{1}{\pi a^2}\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a} = \frac{4a}{3\pi}.$$

The same basic method can be applied to solid bodies, as illustrated in the next example.

Example 5 Find the position of the centre of mass of a uniform, solid, right circular cone of height h.

Once more a diagram is essential; see Fig. 3.6. We consider the cone to be composed of small elements, in this case circular discs of height δx. The disc shown is at a height x above O, the centre of the base. Suppose the radius of the base of the cone is a. By similar triangles

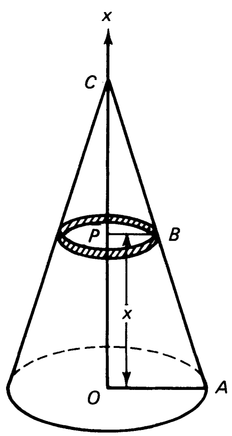


Fig. 3.6

$$\frac{CP}{CO}=\frac{PB}{OA} \quad\Rightarrow\quad \frac{h-x}{h}=\frac{PB}{a} \quad\Rightarrow\quad PB=\frac{a}{h}(h-x).$$

Hence the volume δV of the disc is given by

$$\delta V \approx \pi PB^2 \delta x = (\pi a^2/h^2)(h-x)^2\delta x.$$

From this the volume V of the cone can be obtained by integration. Thus

$$V=\left(\frac{\pi a^2}{h^2}\right)\int_0^h (h-x)^2\,\mathrm{d}x=\left(\frac{\pi a^2}{h^2}\right)\int_0^h (h^2+x^2-2hx)\,\mathrm{d}x$$

$$=\left(\frac{\pi a^2}{h^2}\right)\left[h^2x+\frac{x^3}{3}-hx^2\right]_0^h=\frac{1}{3}\pi a^2 h.$$

Therefore, if m is the mass of the cone, the mass δm of the disc satisfies

$$\delta m/m=\delta V/V \quad\Rightarrow\quad \delta m=(3m/h^3)(h-x)^2\delta x.$$

Suppose the centre of mass of the cone is at a height $\bar{x}$ above O. Since P is at a height x above O, the x-component of (3.3) gives

$$\bar{x}=\frac{\Sigma\delta m\,.\,x}{\Sigma\delta m}=\frac{3}{h^3}\Sigma x(h-x)^2\delta x.$$

Hence, on replacing the summation by integration,

$$\bar{x}=\frac{3}{h^3}\int_0^h x(h-x)^2\,\mathrm{d}x=\frac{3}{h^3}\left[\frac{1}{2}x^2h^2-\frac{2}{3}x^3h+\frac{1}{4}x^4\right]_0^h=\frac{1}{4}h.$$

The final example combines ideas already used in two previous examples, namely Example 2, p. 31, and Example 5.

Example 6 Show by integration that the centre of mass of a uniform, solid hemisphere of radius a is at a distance $(3a/8)$ from the centre of its plane face.

The plane base of a uniform, solid hemisphere of radius a and density 2ρ coincides with the base of a uniform, solid, circular cylinder of radius a, height $2a$ and density ρ. The two solids form one composite body. Show that the centre of mass of this composite body is at a distance $(9a/20)$ from the centre of the plane base of the hemisphere.

We regard the hemisphere as composed of many circular discs of thickness δy, one of which is shown in Fig. 3.7(i). Suppose $ON=y$.

Pythagoras' theorem $\Rightarrow$ $NC^2=OC^2-ON^2$ $\Rightarrow$ $NC=\sqrt{(a^2-y^2)}$. Hence the volume δV of the disc is approximately $\pi(a^2-y^2)\delta y$, and its mass δm is approximately $m\delta V/V$, where $m\ (=\frac{2}{3}\pi\rho a^3)$ and $V=(\frac{2}{3}\pi a^3)$ are the mass and volume of the hemisphere. Thus

$$\delta m \approx (3m/2a^3)(a^2-y^2)\delta y.$$

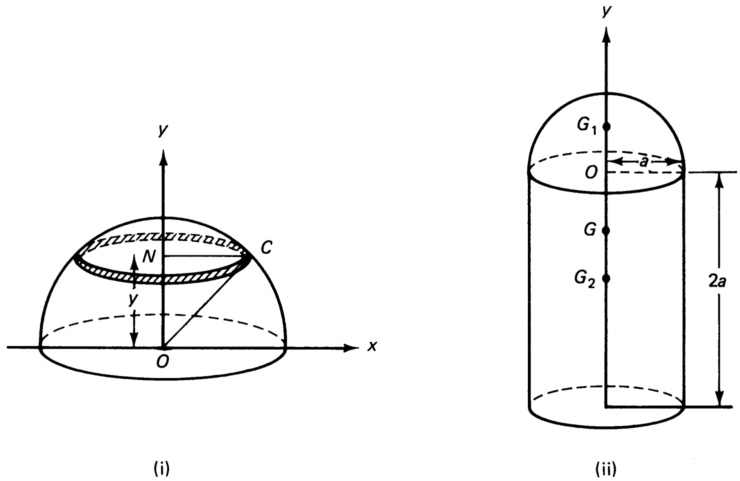


Fig. 3.7

Since the centre of mass of the disc is at a height y above O, the height $\bar{y}$ of the centre of mass of the hemisphere above O satisfies

$$\bar{y} = \left(\frac{3}{2a^3}\right)\int_0^a y(a^2 - y^2)\,\mathrm{d}y = \frac{3a}{8},$$

as required.

The composite solid is shown in Fig. 3.7(ii). The masses of the two bodies are $(2\rho)(\frac{2}{3}\pi a^3) = \frac{4}{3}\pi\rho a^3$ (hemisphere) and $(\rho)(\pi a^2 . 2a) = 2\pi\rho a^3$ (cylinder). The y coordinates of their respective centres of mass G_1 and G_2 are $(3a/8)$ and $(-a)$ respectively. Let the y coordinate of the centre of mass G of the composite solid be $\bar{y}$.

$$(3.3) \quad \Rightarrow \quad \bar{y} = \frac{(\frac{4}{3}\pi\rho a^3)(\frac{3}{8}a) + (2\pi\rho a^3)(-a)}{(\frac{4}{3}\pi\rho a^3 + 2\pi\rho a^3)} = -\frac{9}{20}a.$$

We conclude this chapter by stressing the importance of understanding the principles, not remembering the individual results.

Exercise 3

1 Particles of mass 3 kg, 1 kg and 2 kg have position vectors $(\mathbf{i} + \mathbf{j})$ m, $(-5\mathbf{i} + 4\mathbf{k})$ m and $(-2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ m respectively. Find the position vector of the centre of mass of the three particles.

2 A baby's toy consists of five circular discs threaded onto a light shaft which passes through the centre of each disc such that the discs fit closely together. Each disc is made of uniform plastic and is 1 cm thick. The radii of the discs, in order from the bottom, are 5 cm, 4 cm, 3 cm, 2 cm and 1 cm. Find the position of the centre of mass of the toy.

3 Show that the centre of mass of a triangular lamina of uniform thickness is at the point where the medians of the triangle intersect.

4 Show that the centre of mass of a hemispherical shell of uniform thickness and radius r is on the axis of symmetry at a distance $\frac{1}{2}r$ from the centre.

5 Find the area of the finite region D contained between the x-axis and the curve $y = x(2 - x)^2$, and find the position of the centroid of D.

6 A uniform, solid frustum of a right circular cone has two circular faces of radii $2a$ and a at a distance $2a$ apart. Calculate the distance of the centre of mass of the frustum from the larger circular face. (A *frustum* of a right circular cone is the part of the cone between the base and a plane parallel to the base.)

7 A piece of uniform wire is in the shape of a semicircle of radius R. Show that the centre of mass is at a distance $2R/\pi$ from the centre. Generalize this result by finding the position of the centre of mass of a piece of uniform wire in the shape of an arc of a circle of radius R which subtends an angle 2α at the centre.

8 Sketch on a diagram the region defined by the inequalities

$$0 \leqslant x \leqslant \tfrac{1}{2}\pi, \quad 0 \leqslant y \leqslant \cos x.$$

Find the area of this region and the coordinates of its centroid.

4 Statics

4.1 Introduction

The purpose of this chapter is to study the conditions under which particles and rigid bodies can rest in *equilibrium*, that is without moving. This branch of mechanics is called *statics*.

4.2 The equilibrium of a particle

We are easily able to state a necessary condition for a *particle* to be in equilibrium. A particle which is permanently at rest has zero acceleration. Hence Newton's Second Law with $\mathbf{a} = \mathbf{0}$ becomes

$$\mathbf{F} = \mathbf{0}. \tag{4.1}$$

It needs to be stressed, and remembered, that $\mathbf{F}$ in equation (4.1) is the (vector) sum of all the forces acting on the particle.

Example 1 A particle P of mass m is attached to the ends of two identical elastic strings, each of natural length l and modulus of elasticity λ. The other ends of the strings are fastened to two points Q and R which are at the same level a distance d apart. Show that in equilibrium each string makes an angle θ with the vertical, where $(d\cot\theta - 2l\cos\theta) = (mgl/\lambda)$.

The situation is illustrated in Fig. 4.1. By symmetry the magnitudes of the tensions in the two strings are equal; let this magnitude be T. Unit vectors

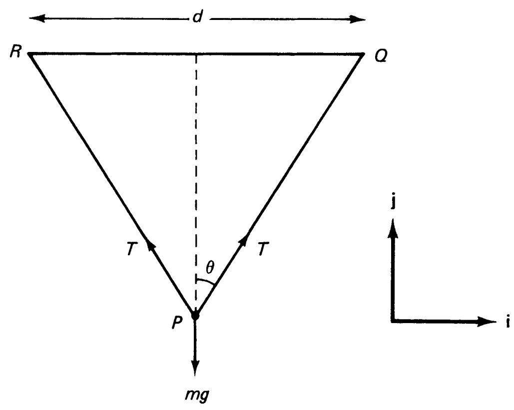


Fig. 4.1

parallel to PQ and PR are $(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)$ and $(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)$ respectively; hence the forces in the strings PQ and PR are $T(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)$ and $T(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)$ respectively. The only other force on P is its weight $-mg\mathbf{j}$.

$$(4.1) \quad \Rightarrow \quad T(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta) + T(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta) - mg\mathbf{j} = \mathbf{0}.$$

$$\Rightarrow \quad 2T\cos\theta = mg \quad \Rightarrow \quad T = \tfrac{1}{2}mg\sec\theta.$$

By Hooke's Law $T = (\lambda/l)(PQ - l)$. Since $PQ = \frac{1}{2}d\operatorname{cosec}\theta$, we find

$$T = \frac{\lambda}{2l}(d\operatorname{cosec}\theta - 2l)$$

$$\Rightarrow \quad \tfrac{1}{2}mg\sec\theta = \frac{\lambda}{2l}(d\operatorname{cosec}\theta - 2l)$$

$$\Rightarrow \quad (d\cot\theta - 2l\cos\theta) = \frac{mgl}{\lambda}.$$

When we consider a system of more than one particle, we have to apply (4.1) to each particle. This is illustrated in the next example.

Example 2 Figure 4.2 shows a light string AC passing over a smooth, small pulley fixed at B. The end A of the string is fastened to a particle of mass m and the end C to a particle of mass M. One end of a second light string is also fastened to the particle of mass M, the other end being fastened to a fixed point D at the same level as B, where $BD = CD$. Given that the system is in equilibrium show that

$$m\cos\theta = M\cos 2\theta,$$

where θ is $\angle ABC$. Obtain the ratio of M to m, and the magnitude S of the tension in CD when $\theta = \pi/6$.

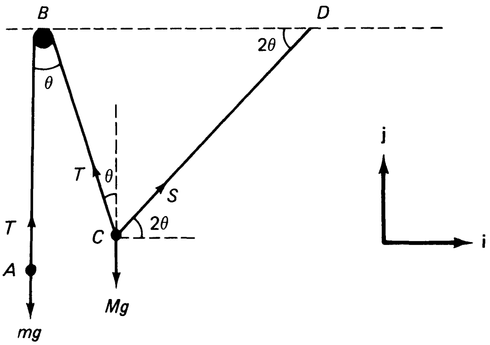


Fig. 4.2

The string is light, so its weight may be neglected. Hence the magnitude T of the tension in AC is the same throughout the string. By applying (4.1) to the particle of mass m at A, we find

$$T = mg.$$

Since $BD = CD$, $\angle BDC = \pi - 2(\angle DBC) = \pi - 2(\frac{1}{2}\pi - \theta) = 2\theta$. Hence unit vectors along $\overrightarrow{CB}$ and $\overrightarrow{CD}$ are $(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)$ and $(\mathbf{i}\cos 2\theta + \mathbf{j}\sin 2\theta)$, where the directions $\mathbf{i}, \mathbf{j}$ are shown on Fig. 4.2. Therefore, applying (4.1) to the particle of mass M,

$$T(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta) + S(\mathbf{i}\cos 2\theta + \mathbf{j}\sin 2\theta) - Mg\mathbf{j} = \mathbf{0}$$

$$\Rightarrow \quad T\sin\theta = S\cos 2\theta, \quad T\cos\theta + S\sin 2\theta = Mg.$$

$$T = mg \quad \Rightarrow \quad S = mg\sin\theta/\cos 2\theta$$

$$\Rightarrow \quad m[\cos\theta + (\sin\theta\sin 2\theta)/\cos 2\theta] = M$$

$$\Rightarrow \quad m\cos\theta = M\cos 2\theta,$$

as required, and using the identity $\cos\theta\cos 2\theta + \sin\theta\sin 2\theta \equiv \cos\theta$.

$$\theta = \frac{\pi}{6} \quad \Rightarrow \quad \cos\theta = \sqrt{3}/2, \quad \cos 2\theta = 1/2 \quad \Rightarrow \quad M : m = \sqrt{3} : 1.$$

$$\theta = \frac{\pi}{6} \quad \Rightarrow \quad \sin\theta = 1/2 = \cos 2\theta \quad \Rightarrow \quad S = T = mg.$$

4.3 Friction

The examples of the previous section illustrate the important point that many common forces have the same form whether we are dealing with dynamics (motion) or statics (no motion). This is particularly true for forces associated with

(i) taut, inextensible strings and rigid rods, and
(ii) elastic strings and springs.

However the case of friction, the force between two rough surfaces in contact, does not fall into this category and needs to be considered separately.

Let us consider a particle of mass m in contact with a rough, plane surface, which need not be horizontal. The surface exerts a force $\mathbf{R}$ on the particle. Let $\mathbf{j}$ be a unit vector perpendicular to, and pointing away from, the surface, as shown in Fig. 4.3. In general $\mathbf{R}$ is not parallel to $\mathbf{j}$. We call the component of $\mathbf{R}$ in the direction of $\mathbf{j}$ the *normal reaction* and denote it by N. Therefore

$$N = \mathbf{R}\,.\,\mathbf{j} = |\mathbf{R}|\cos\theta, \tag{4.2}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{j}$. It is a matter of everyday experience, and common sense, that N cannot be negative. Now define $\mathbf{F}$, and F, by

$$\mathbf{F} = \mathbf{R} - N\mathbf{j}, \quad F = |\mathbf{F}|. \tag{4.3}$$

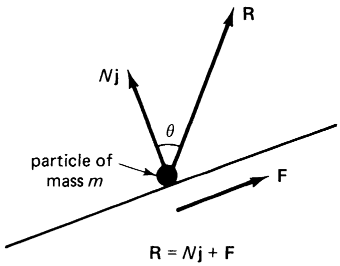


Fig. 4.3

Since N is the component of $\mathbf{R}$ in the direction of $\mathbf{j}$, $\mathbf{F}$ must be perpendicular to $\mathbf{j}$, that is $\mathbf{F}$ is parallel to the surface. Hence

$$F = |\mathbf{F}| = |\mathbf{R}| \sin\theta. \tag{4.4}$$

So far we have not needed to know whether the particle in contact with the surface is moving relative to the surface or not. But from our study of dynamics we know that when the particle is moving relative to the surface, F and N are related by $F = \mu N$, where μ is the *coefficient of friction.* We recall that surfaces for which $\mu = 0$ are said to be smooth, and surfaces for which $\mu > 0$, the usual case, are said to be rough. The value of μ, which is invariably positive but less than 1, depends on the nature of the surfaces in contact. We call $\mathbf{F}$ the *frictional force* (or simply the *friction*) because $\mathbf{F}$ acts in such a way as to oppose motion between the particle and the rough surface.

Now suppose that the particle is *not* moving relative to the surface. In this case experiments show that there is no simple equation, such as $F = \mu N$, relating F and N. But they also show that F cannot exceed μN, that is

$$F \leqslant \mu N. \tag{4.5}$$

This inequality implies that the rough surface can prevent a particle slipping on it, provided that a frictional force $\mathbf{F}$ of magnitude greater than μN is *not* required. The rough surface cannot prevent a particle slipping on it if a value of F greater than μN is required; in such a case slipping occurs with $F = \mu N$ and the friction is said to be *limiting*.

The application of these ideas, particularly (4.5), is illustrated in the following examples.

Example 3 A particle of mass m rests on a rough plane inclined at an angle α to the horizontal. Show that $\alpha \leqslant \lambda$, where λ is the acute angle for which $\tan\lambda = \mu$, where μ is the coefficient of friction.

The situation is illustrated in Fig. 4.4. The particle is acted on by its weight $-mg(\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha)$ and the reaction of the surface $N\mathbf{j} + \mathbf{F}$.

$$(4.1) \quad \Rightarrow \quad \mathbf{F} + N\mathbf{j} - (mg\sin\alpha)\mathbf{i} - (mg\cos\alpha)\mathbf{j} = \mathbf{0}.$$

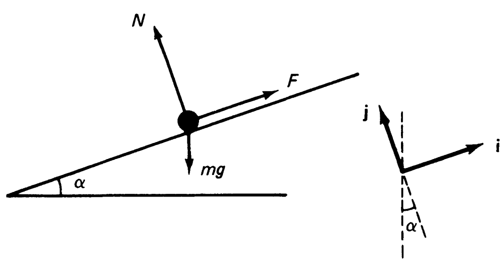


Fig. 4.4

Since $\mathbf{F}$ is perpendicular to $\mathbf{j}$, we must have $\mathbf{F} = F\mathbf{i}$. Hence

$$F = mg\sin\alpha, \quad N = mg\cos\alpha.$$

$$(4.5) \quad \Rightarrow \quad mg\sin\alpha \leqslant \mu mg\cos\alpha \quad \Rightarrow \quad \tan\alpha \leqslant \mu$$

$$\Rightarrow \quad \alpha \leqslant \lambda,$$

as required.

In this example, the component of the particle's weight down the plane, that is $mg\sin\alpha$, would tend to make the particle slide down the plane; however the roughness of the plane prevents this movement, by generating a frictional force of magnitude $mg\sin\alpha$, only if $\alpha \leqslant \lambda$. The angle λ for which

$$\tan\lambda = \mu \tag{4.6}$$

is known as the *angle of friction*. This example shows that λ is the angle of greatest slope of a rough plane on which a particle can rest. When the slope α of the plane is greater than λ the particle slides down the plane with $F = \mu N$. From Newton's Second Law it follows that its acceleration is then directed down the slope with magnitude

$$g(\sin\alpha - \mu\cos\alpha) = g\cos\alpha(\tan\alpha - \tan\lambda).$$

Example 4 Two small rings A and B, of mass $2m$ and $3m$ respectively, can slide on a fixed, rough horizontal rod. The coefficient of friction between each ring and the rod is μ. A small smooth ring C of mass $4m$ is threaded on an inextensible string of length $2a$, the ends of which are attached to A and B. The system is in equilibrium. Show that the friction at B cannot be limiting, and determine, in terms of μ and a, the greatest possible value of the distance AB.

The essential sketch is given as Fig. 4.5, which also shows the directions chosen for the unit vectors $\mathbf{i}$ and $\mathbf{j}$. We note that, since the ring C is smooth, the magnitude T of the tension throughout the string is the same. Let AB and BC make angles α and β respectively with the vertical. Applying (4.1) to the ring C gives

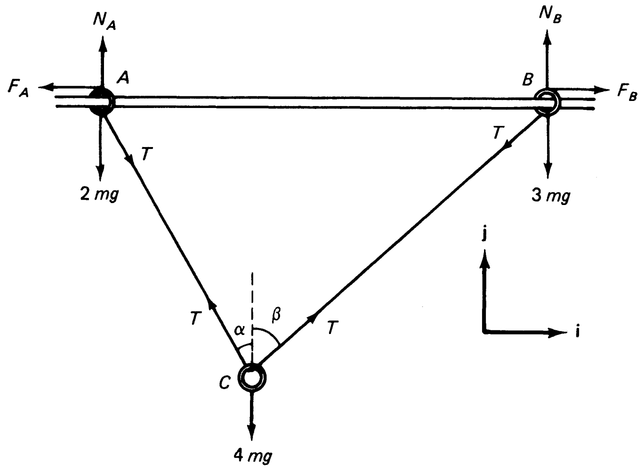


Fig. 4.5

$$T(-\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha) + T(\mathbf{i}\sin\beta + \mathbf{j}\cos\beta) - 4mg\mathbf{j} = \mathbf{0}$$

$$\Rightarrow \quad T\sin\alpha = T\sin\beta, \quad T(\cos\alpha + \cos\beta) = 4mg$$

$$\Rightarrow \quad \alpha = \beta, \quad T = 2mg\sec\alpha = 2mg\sec\beta.$$

Without friction, that is if the rod were smooth, the rings A and B would move towards one another because of the horizontal components of the tension in the string. Hence the frictional forces on A and B act in the directions shown, trying to oppose such motions.

$$\text{(4.1) applied to } A \quad \Rightarrow \quad T(\mathbf{i}\sin\alpha - \mathbf{j}\cos\alpha) - 2mg\mathbf{j} + (-F_A\mathbf{i} + N_A\mathbf{j}) = \mathbf{0}$$

$$\Rightarrow \quad T\sin\alpha = F_A, \quad T\cos\alpha + 2mg = N_A.$$

$$T = 2mg\sec\alpha \quad \Rightarrow \quad F_A = 2mg\tan\alpha, \quad N_A = 4mg.$$

$$\text{(4.1) applied to } B \quad \Rightarrow \quad T(-\mathbf{i}\sin\beta - \mathbf{j}\cos\beta) - 3mg\mathbf{j} + (F_B\mathbf{i} + N_B\mathbf{j}) = \mathbf{0}$$

$$\Rightarrow \quad T\sin\beta = F_B, \quad T\cos\beta + 3mg = N_B.$$

$$T = 2mg\sec\beta, \quad \alpha = \beta \quad \Rightarrow \quad F_B = 2mg\tan\alpha, \quad N_B = 5mg.$$

For equilibrium to be possible, (4.5) must be satisfied at *both* A and B.

$$\text{(4.5) applied to } A \quad \Rightarrow \quad 2mg\tan\alpha \leqslant 4\mu mg \quad \Rightarrow \quad \tan\alpha \leqslant 2\mu. \qquad (4.7)$$

$$\text{(4.5) applied to } B \quad \Rightarrow \quad 2mg\tan\alpha \leqslant 5\mu mg \quad \Rightarrow \quad \tan\alpha \leqslant 5\mu/2. \qquad (4.8)$$

Since both (4.7) and (4.8) must be satisfied, and since $2\mu < 5\mu/2$, it follows that $\tan\alpha \leqslant 2\mu$. Hence $F_B \leqslant 4\mu mg < \mu N_B$, that is the friction at B is not limiting.

$$\alpha = \beta \quad \Rightarrow \quad AB = 2a \sin \alpha.$$

$$\tan \alpha \leqslant 2\mu \quad \Rightarrow \quad \sin \alpha \leqslant 2\mu/\surd(1 + 4\mu^2)$$

$$\Rightarrow \quad AB \leqslant \frac{4\mu a}{\surd(4\mu^2 + 1)}.$$

Therefore the greatest possible value for AB is $4\mu a/\surd(4\mu^2 + 1)$.

4.4 The equilibrium of a rigid body: I The moment of a force about a fixed axis

There is an essential difference between particles and rigid bodies as far as the conditions for equilibrium are concerned. This difference is that rigid bodies can *turn*.

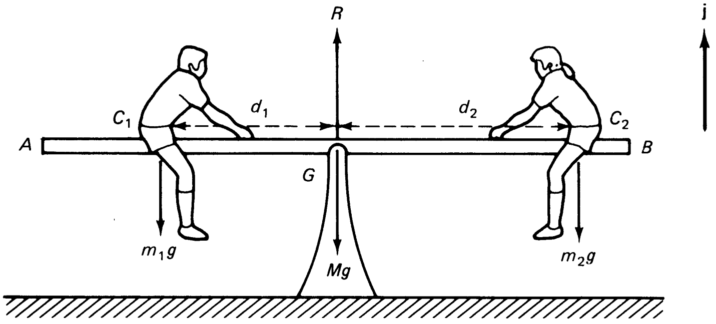


Fig. 4.6

Consider the see-saw shown in Fig. 4.6. This consists of a uniform plank AB which is free to turn about a smooth, horizontal pivot at its centre of mass G. The pivot is held in a fixed position by a support mounted on the ground. Suppose two children of masses m_1 and m_2 sit on the see-saw so that their centres of mass C_1 and C_2 are at horizontal distance d_1 and d_2 from G, and that the see-saw is in balance. Consider the forces acting on the see-saw. Neither child moves; therefore, by (4.1), the see-saw exerts an upward force on each child equal in magnitude to his weight. Consequently, by Newton's Third Law, each child exerts a downward force on the see-saw. Hence the see-saw experiences forces $-m_1g\mathbf{j}$ and $-m_2g\mathbf{j}$ acting through C_1 and C_2 respectively, where $\mathbf{j}$ is a unit vector pointing vertically upwards, as shown in Fig. 4.6. In addition the see-saw is acted on by its own weight $-Mg\mathbf{j}$ (where M is the mass of the see-saw) and the reaction of the pivot. The reaction of the pivot is a force vertically upwards; let it be $R\mathbf{j}$. Therefore the net force $\mathbf{F}$ acting on the see-saw is

$$\mathbf{F} = (R - m_1g - m_2g - Mg)\mathbf{j}.$$

Experiments show, and we shall provide theoretical justification later in §5.4, that, even though the see-saw is not a particle, equation (4.1) must apply for it to be in equilibrium:

$$\Rightarrow \quad R = (m_1 + m_2 + M)g. \tag{4.9}$$

However, and this is where a rigid body such as the see-saw differs from a particle, equation (4.9) is not a *sufficient* condition for equilibrium. Even when (4.9) is satisfied, the see-saw may not be at rest because it may be turning, that is rotating, about G. Experience shows that turning does not occur if the distances d_1 and d_2 are such that

$$m_1 d_1 - m_2 d_2 = 0. \tag{4.10}$$

Thus if the first child is, for example, twice as heavy as the second child, so that $m_1 = 2m_2$, turning of the see-saw about G will occur unless the second child is twice as far from G as the first, that is unless $d_2 = 2d_1$.

Provided equations (4.9) and (4.10) are both satisfied, the see-saw is at rest. In other words (4.9) and (4.10) are *necessary and sufficient conditions* for the see-saw to be in equilibrium.

Equation (4.10) is a consequence of a basic principle of statics. To state this principle we need to define the moment of a force. The proper treatment of this concept requires the use of the vector product, and the moment of a force is defined in this way in many books on vectors and mechanics. Here we content ourselves with a restricted definition, but this means that we can only deal with cases in which all the forces acting on the rigid body lie in a plane Π. Because of this the body can turn about axes perpendicular to Π. Suppose one such axis passes through the point P in Fig. 4.7, where Π is the plane of the paper, and let a force $\mathbf{F}$ of magnitude $F(= |\mathbf{F}|)$ act in the direction shown. Then the *moment* of the force $\mathbf{F}$ in the *anti-clockwise sense* about an *axis* through P is defined to be L, where

$$L = Fp, \tag{4.11}$$

and p is the *perpendicular* distance of P from the *line of action* of $\mathbf{F}$. Other symbols, particularly G and N, are often used instead of L.

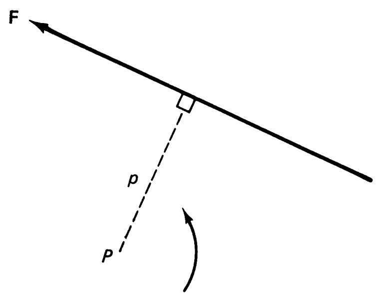


Fig. 4.7

The magnitude of L measures the effectiveness of $\mathbf{F}$ in turning the body on which it acts, in an anti-clockwise sense, about an axis through P. Consider once more the see-saw shown in Fig. 4.6. The force $-m_1 g\mathbf{j}$ through C_1 tends

to turn the see-saw in an anti-clockwise sense about G. According to (4.11) the moment L_1 of this force satisfies

$$L_1 = |-m_1g\mathbf{j}| \times d_1 = m_1gd_1. \tag{4.12}$$

Naturally other forces acting on a body may tend to turn it in a *clockwise sense* about an axis. For such a force, we define its moment to be $-Fp$, where F and p have the same meanings as they do in (4.11). Thus forces tending to turn a body in a clockwise sense about an axis have negative moments. Hence the force $-m_2g\mathbf{j}$ acting through C_2 on the see-saw in Fig. 4.6 has a moment L_2 about G, where

$$L_2 = -|-m_2g\mathbf{j}| \times d_2 = -m_2gd_2. \tag{4.13}$$

The *total moment* about an axis of all the forces acting on a body is defined to be the sum of the moments about this axis of the individual forces, with anti-clockwise moments being positive and clockwise moments being negative, according to the rules given above. We illustrate this definition for the see-saw. Apart from the forces acting through C_1 and C_2 which give rise to the moments L_1 and L_2 in (4.12) and (4.13) respectively, the other forces acting on the see-saw are its weight $-Mg\mathbf{j}$ and the reaction of the pivot $R\mathbf{j}$. However both of these forces act through G; hence the moment of each about G is zero since, for each force, $p = 0$ in (4.11). Therefore the total moment L about G is given by

$$L = L_1 + L_2 = (m_1d_1 - m_2d_2)g, \tag{4.14}$$

where (4.12) and (4.13) have been used to obtain the values of L_1 and L_2. For the particular case of the see-saw, the total moment L about G in (4.14) is zero because of (4.10).

In the general case of a rigid body acted on by forces which all lie in a plane Π, the *necessary and sufficient conditions* for equilibrium are

$$\mathbf{F} = \mathbf{0}, \quad L = 0, \tag{4.15}$$

where $\mathbf{F}$ is the sum of all the forces acting on the body and L is the total moment of all the forces about *any* axis perpendicular to Π.

Before illustrating the application of (4.15), it is appropriate to stress the word 'any' in the previous sentence. The point is that when $\mathbf{F} = \mathbf{0}$ and the total moment of the forces about one axis is zero, then the total moment about every axis is zero. Once more, consider the see-saw in Fig. 4.6. Let us calculate the total moment L' of all the forces about C_1. Then

$$L' = -Mgd_1 - m_2g(d_1 + d_2) + Rd_1,$$

since the force $-m_1g\mathbf{j}$ has no moment about C_1. Incorporating (4.9),

$$L' = -Mgd_1 - m_2g(d_1 + d_2) + (m_1 + m_2 + M)gd_1 = (m_1d_1 - m_2d_2)g.$$

Hence $L' = L = 0$, according to (4.14). It may be shown similarly that the total moment of the forces about any other axis, such as one through C_2 or through A, is also zero.

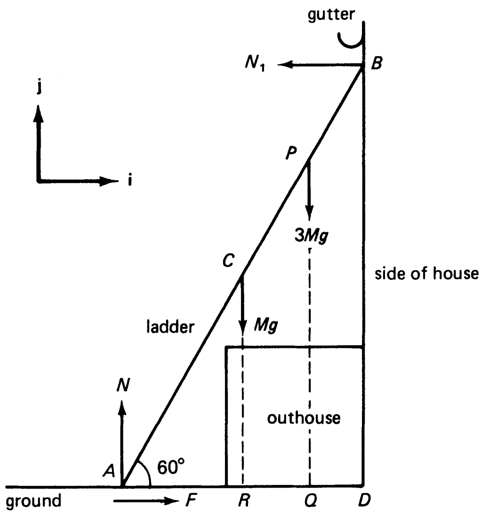


Fig. 4.8

Example 5 A man is using a uniform ladder to paint the outside of his house. His mass is three times that of the ladder. In one place, shown in Fig. 4.8, the ladder cannot be inclined at an angle greater than 60° to the horizontal because of an outhouse. At this inclination the ladder is just not in contact with the outhouse. The man finds that he can reach the gutter only by standing on the top rung of the ladder, whose distance from the top of the ladder is so small that it can be ignored. Given that the side of the house is smooth, show that the coefficient of friction between the ladder and the ground is greater than 0·505.

Before the man is able to complete painting this part of the gutter, a shower causes the ground to become muddy so that the coefficient of friction between the ladder and the ground is reduced to 0·4. Unfortunately the man tries to resume painting without realizing this. How far is he up the ladder before it slips?

Let l be the length of the ladder. Figure 4.8 illustrates the situation when the man is at P, and we shall let the length of AP be rl so that, for example, when the man is at the top of the ladder $r = 1$. The forces acting on the ladder are its weight $-Mg\mathbf{j}$ through C, where $AC = \frac{1}{2}l$ because the ladder is uniform, the reaction of the ground $F\mathbf{i} + N\mathbf{j}$ through A, the reaction of the side of the house $-N_1\mathbf{i}$ through B and the weight of the man $-3Mg\mathbf{j}$ through P. We now apply both conditions in (4.15). The first of these gives

$$(-Mg\mathbf{j} + F\mathbf{i} + N\mathbf{j} - N_1\mathbf{i} - 3Mg\mathbf{j}) = \mathbf{0}$$

$$\Rightarrow \quad (F - N_1)\mathbf{i} + (N - 4Mg)\mathbf{j} = \mathbf{0}$$

$$\Rightarrow \quad F = N_1, \quad N = 4Mg. \tag{4.16}$$

We shall find the total moment L of the forces about A, although, as explained earlier, any point would do equally well. The moment of the weight of the ladder about A is in the clockwise sense, hence negative, and of magnitude $Mg \times AR = Mg(\frac{1}{2}l\cos 60°) = \frac{1}{4}Mgl$, since AR is the perpendicular distance from A to the line of action of the force. Proceeding similarly for all forces, we find

$$\begin{aligned} L = 0 \quad &\Rightarrow \quad -\tfrac{1}{4}Mgl + N_1 \,.\, BD - 3Mg \,.\, AQ = 0 \\ &\Rightarrow \quad -\tfrac{1}{4}Mgl + N_1 \,.\, l\sqrt{3}/2 - 3Mgrl/2 = 0 \\ &\Rightarrow \quad 2\sqrt{3}N_1 l = Mgl(6r + 1) \\ &\Rightarrow \quad N_1 = Mg(6r + 1)/2\sqrt{3}. \end{aligned} \tag{4.17}$$

Equilibrium requires both (4.16) and (4.17) to be satisfied. However, by (4.5), F cannot exceed μN, where $F = N_1$ and $N = 4Mg$. Hence, substituting the value of N_1,

$$\begin{aligned} (4.5) \quad &\Rightarrow \quad Mg(6r + 1)/2\sqrt{3} \leqslant 4\mu Mg \\ &\Rightarrow \quad (6r + 1) \leqslant 8\sqrt{3}\mu. \end{aligned} \tag{4.18}$$

We are told that (4.18) can be satisfied with $r = 1$ when the ground is dry

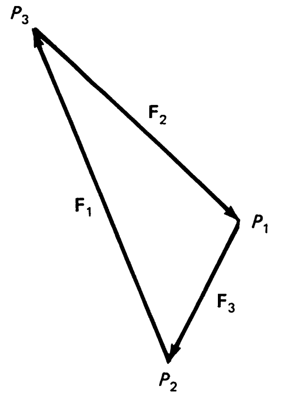
$$\Rightarrow \quad \mu \geqslant 7/(8\sqrt{3}) \approx 0{\cdot}5052.$$

Hence, as required, μ must be greater than $0{\cdot}505$.

When the ground is muddy μ is reduced to $0{\cdot}4$.

$$(4.18) \quad \Rightarrow \quad (6r + 1) \leqslant 8\sqrt{3} \times 0{\cdot}4 \approx 5{\cdot}5426.$$

Hence $r \leqslant 0{\cdot}757$ to three decimal places. So the ladder slips when the man is just over three-quarters of the way up it.



$$\overrightarrow{P_2P_3} = \overrightarrow{P_2P_1} + \overrightarrow{P_1P_3} = -(\overrightarrow{P_3P_1} + \overrightarrow{P_1P_2})$$

Fig. 4.9

The reader is invited to show that the same results are obtained in Example 5 if the total moment of the forces about a point other than A is determined and equated to zero. One could, for example, choose B or C or D.

We conclude this section with a brief remark about dimensions and units. According to (4.11), the dimension of the moment L is $\mathrm{ML^2T^{-2}}$ so that its units are $\mathrm{kg\,m^2s^{-2}}$. These are the same dimensions and units as work and kinetic energy, but when considering a moment we do not use the special unit of 1 J for a moment of 1 $\mathrm{kg\,m^2s^{-2}}$. This unit is reserved exclusively for work and energy. The reason for this is that, although moment on the one hand and work and energy on the other hand have the same units, they are totally different physical quantities. Indeed, as indicated earlier, a proper definition of moment shows it to be a vector, whereas work and energy are scalars.

4.5 The equilibrium of a rigid body: II A body acted on by three forces

An interesting special case occurs when a rigid body is acted on by exactly *three* forces, no two of which are parallel. The conditions for equilibrium given in (4.15) then have a simple geometrical meaning which is often useful.

Suppose the three forces are $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$. By the first condition in (4.15)

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0} \quad \Rightarrow \quad \mathbf{F}_1 = -(\mathbf{F}_2 + \mathbf{F}_3).$$

It then follows from the *triangle law* of vector addition that if, using a suitable scale (e.g. a line segment of 1 cm represents a force of magnitude 1 N), the directed line segments $\overrightarrow{P_3P_1}$ and $\overrightarrow{P_1P_2}$ represent $\mathbf{F}_2$ and $\mathbf{F}_3$, then $\mathbf{F}_1$ is represented by the directed line segment $\overrightarrow{P_2P_3}$. This is illustrated in Fig. 4.9. The triangle $P_1P_2P_3$ is known as a *triangle of forces*, and we have shown that a necessary condition for a rigid body to be in equilibrium under three non-parallel forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ is that these forces can be represented in magnitude and direction by the directed line segments $\overrightarrow{P_2P_3}$, $\overrightarrow{P_3P_1}$, $\overrightarrow{P_1P_2}$ forming the sides of a triangle $P_1P_2P_3$. (We may note in passing that a similar argument may be applied to a body in equilibrium under n non-parallel forces, where $n > 3$; there then exists a *polygon of forces* such that the forces are represented in magnitude and direction by the n sides of the polygon.)

Let us consider the second condition in (4.15), namely $L = 0$. We now have to take account of the lines of action of the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$; let these be l_1, l_2 and l_3 respectively. Since we are still considering only cases in which the forces lie in a plane and are not parallel, it follows that the three lines of action meet one another. Suppose l_2 and l_3 meet at A. Let us calculate the total moment L of the forces about A, which must be zero for the body to be in equilibrium. Since l_2 and l_3 both pass through A, $\mathbf{F}_2$ and $\mathbf{F}_3$ have no moment about A. Hence the total moment L can be zero only if $\mathbf{F}_1$ also has no moment about A; therefore l_1 must also pass through A and the lines of action of the three forces are *concurrent*.

We conclude that the necessary and sufficient conditions for a rigid body to

be in equilibrium under three, coplanar, non-parallel forces are (i) that their lines of action are concurrent, and (ii) that the forces can be represented in magnitude and direction by the sides of a triangle.

The next example shows how useful these conditions can be in simplifying the solution of certain problems.

Example 6 Figure 4.10 shows a non-uniform rod AB of length a and mass m which rests in equilibrium making an angle β with the horizontal. The end A of the rod rests on a rough horizontal table and its centre of mass is at G, where $AG = \frac{2}{3}a$. Equilibrium is maintained by a horizontal string attached to B. Show that the coefficient of friction μ between the rod and the table cannot be less than $\frac{2}{3}\cot\beta$.

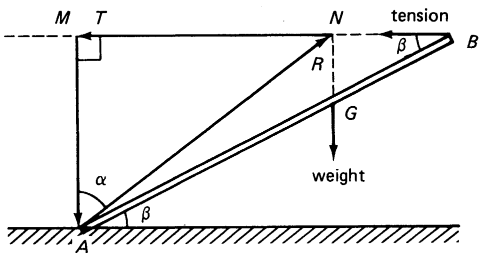


Fig. 4.10

There are three forces acting on the rod: its weight acting vertically downwards through G, the tension in the string acting horizontally through B, and the reaction at A. The lines of action of the first two of these meet at N; hence the line of action of the reaction at A also passes through N.

Let us choose a scale such that $\overrightarrow{AN}$ represents the reaction at A. Since the tension in the string is horizontal, its direction is parallel to $\overrightarrow{NM}$. The weight of the rod is parallel to $\overrightarrow{MA}$. Hence $\triangle ANM$ is a triangle of forces, with $\overrightarrow{NM}$ and $\overrightarrow{MA}$ representing the tension in the string and the weight of the rod respectively. Let $\angle MAN$ be denoted by α.

$$\text{Similar triangles} \;\Rightarrow\; MB/NB = AB/GB = 3$$

$$\Rightarrow\; MB = 3MN/2.$$

$$\cot\beta = MB/MA \;\Rightarrow\; MN = (\tfrac{2}{3}\cot\beta)MA.$$

$$\tan\alpha = MN/MA \;\Rightarrow\; \tan\alpha = \tfrac{2}{3}\cot\beta.$$

Let R, S, F denote the magnitudes of the reaction at A, the normal reaction and friction respectively, so that, using (4.2) and (4.4), $S = R\cos\alpha$, $F = R\sin\alpha$. Then

$$(4.5) \;\Rightarrow\; \mu \geqslant F/S = \tan\alpha \;\Rightarrow\; \mu \geqslant \tfrac{2}{3}\cot\beta,$$

as required. The argument here is very similar to that in Example 3, p. 41. If needed, we can easily find the values of R and T (the magnitude of the tension in the string). Since the tension in the string is represented by $\overrightarrow{NM}$ and the weight of the rod by $\overrightarrow{MA}$, we have

$$T/mg = MN/MA = \tan\alpha, \quad T = \tfrac{2}{3}mg\cot\beta.$$

Of course, the results obtained by geometry in Example 6 could also have been obtained by the methods used in §4.4 and illustrated in Example 5. The reader is advised to attempt Example 6 using these methods, and to compare the length and ease of the solution with that just given.

Exercise 4

1 Two light, elastic strings AB and CD have the same natural length l and would be doubled in length by tensions of magnitudes $2W$ and $3W$ respectively. The ends B and C are joined together, the end A is fixed, and at D a particle of mass W/g hangs at rest with both strings vertical. Find the distance AD.

2 The ends A and E of a light, inextensible string $ABCDE$, where $AB = DE$ and $BC = CD$, are attached to two fixed points at the same horizontal level. Particles of mass m, $3m$ and m are attached to B, C and D respectively and hang in equilibrium with AB and BC making acute angles θ and ϕ respectively with the horizontal. By finding in terms of m, θ and ϕ the magnitudes of the tensions in AB and BC, show that

$$5\tan\phi = 3\tan\theta.$$

3 A particle of mass M rests on a rough plane inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}8$. The particle is attached to one end of a light, inextensible string which lies along a line of greatest slope of the plane and passes over a small, smooth pulley at the top of the plane. Hanging freely from the other end of the string is a particle of mass m. If the largest value of m for which equilibrium is possible is M, find the smallest such value.

4 Two particles of masses m_1 and m_2 are attached to the ends A and B respectively of a light, inextensible string which passes over a smooth pulley at the top C of a fixed rough plane inclined at an angle α to the horizontal. The particles are at rest with m_1 in contact with the plane and m_2 hanging freely, so that AC is along a line of greatest slope of the plane and BC is vertical. Show that

$$m_1(\sin\alpha + \mu\cos\alpha) \geqslant m_2 \geqslant m_1(\sin\alpha - \mu\cos\alpha),$$

where μ is the coefficient of friction. Experiments are conducted with a set of inclined planes made of identical material. Find μ, given that m_1 is on the point of moving down a plane for which $\alpha = 60°$, and on the point of moving up a plane for which $\alpha = 30°$.

5 A uniform ladder of mass m and length $2l$ rests with its upper end against a smooth, vertical wall. Its lower end stands on rough, horizontal ground, and the coefficient of friction between the ladder and the ground is $\frac{1}{2}$. Given that the ladder is in limiting equilibrium, find the angle it makes with the horizontal. A man of mass m climbs the ladder. How far up can he go before slipping occurs?

6 A uniform solid is made by joining together two solid, right circular cones, C_1 of height h_1 and C_2 of height h_2 (where $h_2 > h_1$), each with base radius a, so that their bases coincide. Show that the centre of mass of the solid lies in the cone C_2 at a distance $\frac{1}{4}(h_2 - h_1)$ from the common base. Prove that the solid can rest in equilibrium on a

horizontal table with the curved surface of C_1 touching the table if

$$h_2 < h_1 + (4a^2/h_1).$$

7 A smooth, uniform sphere of radius $3a$ and mass M rests against a smooth, vertical wall. The sphere is supported in equilibrium by a light, inextensible string of length $2a$, one end of which is attached to a point on the surface of the sphere. The other end of the string is attached to a point on the wall. Determine the magnitude of the tension in the string.

8 A uniform rod AB of length $4a$ rests in a vertical plane with its end A on a rough, horizontal plane and a point C of the rod in contact with a smooth peg, where $AC = 3a$. In equilibrium the rod makes an angle α with the horizontal. Show that the coefficient of friction between the rod and the plane cannot be less than

$$\sin 2\alpha/(2 - \cos 2\alpha).$$

9 A uniform, circular hoop of mass M hangs over a rough, horizontal peg A. The hoop is pulled with a gradually increasing horizontal force of magnitude P. This force is applied at the other end B of the diameter through A and acts in the vertical plane of the hoop. Given that the system is in equilibrium when AB is inclined at an angle β to the downward vertical, show that the ratio of the frictional force to the normal reaction at the peg is $(\tan \beta)/(2 + \tan^2 \beta)$, and deduce that, when the coefficient of friction is $\frac{1}{2}$, the hoop never slips however hard it is pulled.

10 Two uniform, straight rods AB and BC of equal weight but unequal length are smoothly jointed at B. The rod AB is smoothly pivoted at a fixed point A above a rough, horizontal plane with which the end C of the rod BC is in contact. The rods rest in equilibrium in a vertical plane through A when equally inclined at an angle α to the horizontal, with B above the level of A. Show that the coefficient of friction at C is at least $\frac{1}{2}\cot \alpha$.

11 Two uniform rods AB and BC, each of mass per unit length m, have lengths $3a$ and $5a$ respectively, and are smoothly jointed together at B. The system is in equilibrium with C resting on a rough, horizontal floor at a distance $6a$ from a rough, vertical wall, with which A is in contact. The rod AB is horizontal and the plane of the rods is perpendicular to the floor and the wall.

(a) Find the height of AB above the floor.

(b) Find the frictional force exerted by the wall at A.

(c) Find the horizontal and vertical components of the force exerted on BC at C.

(d) Show that the coefficient of friction at C must be at least $6/13$, and find the least possible value of the coefficient of friction at A.

12 A uniform heavy sphere of weight of magnitude W is kept in equilibrium on a rough plane, inclined to the horizontal at an acute angle γ, by a light horizontal string attached to the highest point of the sphere. The string lies in a vertical plane containing a line of greatest slope of the incline. Find the magnitude of the tension in the string and show that the coefficient of friction between the sphere and the plane cannot be less than $\tan(\gamma/2)$.

13 A uniform ladder of length $2a$ and mass M is leaning against a smooth, vertical wall with its foot on rough, horizontal ground; its angle of inclination to the horizontal is α and the coefficient of friction between the foot of the ladder and the ground is μ. A man of mass λM climbs the ladder, and friction becomes limiting when he is a fraction σ of the way up the ladder. Find σ in terms of λ, μ and α. Given that friction becomes limiting when a man of mass $5M$ is four-fifths of the way up the ladder, deduce that $\sigma = (3\lambda + 1)/(4\lambda)$. Hence show that a man of mass less than M can climb safely right to the top of the ladder, and that no man is in any danger of slipping until he is three-quarters of the way up.

5 Coplanar forces acting on a rigid body

5.1 Introduction

In Chapter 4 we discussed the conditions which ensure that a rigid body is in equilibrium under a set of coplanar forces. In the present chapter we consider sets of coplanar forces acting on a rigid body which may not be in equilibrium. This work

(i) provides justification for some of the results stated, without proof, in Chapters 3 and 4;

(ii) prepares the ground for the study of the motion of rigid bodies in Chapter 6.

5.2 Some preliminary results involving couples

We begin with some preliminary results, the first of which introduces the important concept of a couple. Consider a force **F** with line of action l, and a force $-\mathbf{F}$ with line of action m, as shown in Fig. 5.1. Such a system of two forces equal in magnitude but opposite in direction is known as a *couple*. Let us calculate the total moments L_1, L_2 and L_3 of this couple about the points P_1, P_2 and P_3 respectively, as shown in Fig. 5.1. We draw lines perpendicular to l and m through P_1, P_2 and P_3 which meet l in the points M_1, M_2 and M_3 and m in the points N_1, N_2 and N_3.

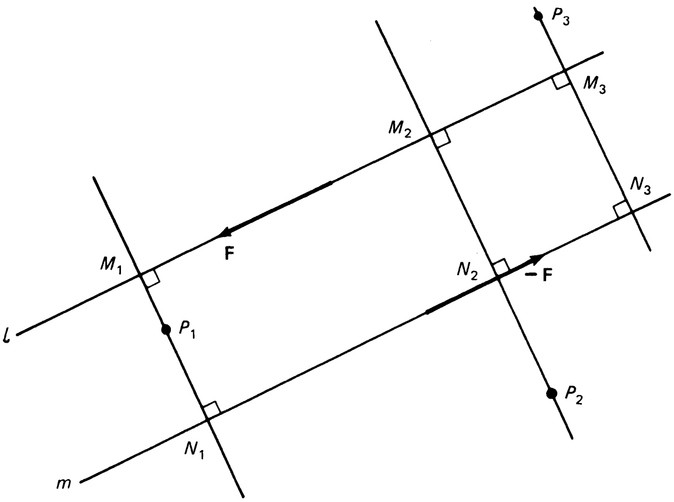


Fig. 5.1

Both $\mathbf{F}$ and $-\mathbf{F}$ have anti-clockwise, hence positive, moments about P_1 and so, by the basic definition in (4.11),

$$L_1 = |\mathbf{F}|M_1P_1 + |-\mathbf{F}|P_1N_1 = |\mathbf{F}|(M_1P_1 + P_1N_1) = |\mathbf{F}|M_1N_1. \qquad (5.1)$$

On the other hand, while $\mathbf{F}$ has an anti-clockwise moment about P_2, the force $-\mathbf{F}$ has a clockwise, hence negative, moment about P_2. Thus

$$L_2 = |\mathbf{F}|M_2P_2 - |-\mathbf{F}|N_2P_2 = |\mathbf{F}|(M_2P_2 - N_2P_2) = |\mathbf{F}|M_2N_2.$$

Similarly we obtain

$$L_3 = -|\mathbf{F}|M_3P_3 + |-\mathbf{F}|N_3P_3 = |\mathbf{F}|(N_3P_3 - M_3P_3) = |\mathbf{F}|M_3N_3.$$

But, as is clear from Fig. 5.1, $M_1N_1 = M_2N_2 = M_3N_3$. Hence

$$L_1 = L_2 = L_3. \qquad (5.2)$$

Since P_1, P_2 and P_3 are arbitrary points, we deduce that the total moment of a couple is the same about every point in the plane in which the forces act; hence we can refer to the *moment of a couple* without specifying the point about which the moment is taken.

Example 1 A uniform rigid beam AB of length $6a$ and mass m rests on supports P and Q at the same level, where $AP = a$, $PQ = 3a$, $QB = 2a$. Calculate the magnitudes of the forces exerted by the beam on the supports.

When a couple of moment K is applied to the beam, the forces exerted by the beam on the supports are equal. Find K.

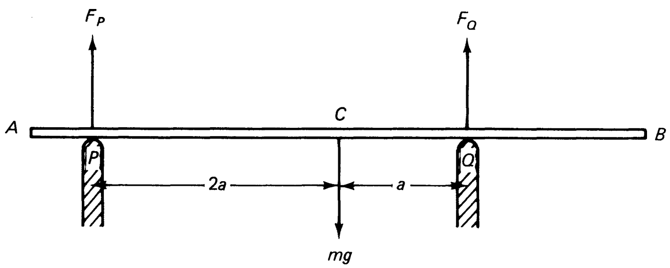


Fig. 5.2

The situation is illustrated in Fig. 5.2. By Newton's Third Law the forces exerted by the beam on the supports are equal and opposite to those exerted on the beam by the supports. The forces exerted on the beam by the supports are clearly vertical; let their magnitudes be F_P and F_Q. We use (4.15) with L being the moment about P.

$$\mathbf{F} = \mathbf{0} \quad \Rightarrow \quad F_P + F_Q = mg.$$

$$L = 0 \quad \Rightarrow \quad F_Q \, . \, 3a - mg \, . \, 2a = 0.$$

Hence $F_P = \frac{1}{3}mg$, $F_Q = \frac{2}{3}mg$. When the couple acts, $F_P = F_Q = \frac{1}{2}mg$ to satisfy $\mathbf{F} = \mathbf{0}$. Hence the reaction on the beam at Q is less when the couple acts. Hence K must be positive. For equilibrium, the total moment about P must be zero. Therefore

$$K + \tfrac{1}{2}mg\,.\,3a - mg\,.\,2a = 0.$$

Hence $K = \frac{1}{2}mga$.

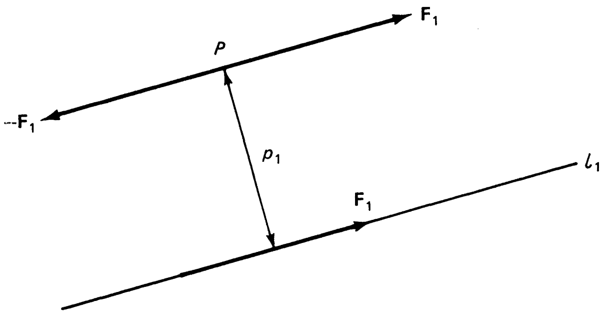


Fig. 5.3

Next we consider a single force $\mathbf{F}_1$ with line of action l_1. Suppose $\mathbf{F}_1$ is acting on a rigid body. Let P be any point whatsoever. Clearly the effect of $\mathbf{F}_1$ on the rigid body is unchanged if we suppose that, in addition to $\mathbf{F}_1$ itself, forces $\mathbf{F}_1$ and $-\mathbf{F}_1$ act at P, for these two forces acting at P have zero sum and zero moment about any point. However the original force $\mathbf{F}_1$ acting along l_1, and the force $-\mathbf{F}_1$ acting at P form a couple. For the situation shown in Fig. 5.3, the moment L_1 of this couple is positive and equal to $|\mathbf{F}_1|p_1$, where p_1 is the perpendicular distance from P to l_1. We conclude that a single force $\mathbf{F}_1$ with line of action l_1 is equivalent, in its effect on a rigid body, to a force $\mathbf{F}_1$ acting at an arbitrary point P of the body together with a couple of moment L_1.

We are now in a position to consider the general case of a rigid body acted on by n coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ with lines of action $l_1, l_2, \ldots, l_n$. Let P be any point of the body. From the preceding paragraph, it follows that each force $\mathbf{F}_r$ is equivalent to a force $\mathbf{F}_r$ acting through P together with a couple of moment L_r. Hence the complete system of n forces is equivalent to a force $\mathbf{F}$ acting through P, where

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n, \tag{5.3}$$

together with a couple of moment L, where

$$L = L_1 + L_2 + \ldots + L_n. \tag{5.4}$$

We can illustrate this theoretical result by establishing a result that was assumed without proof in Chapters 3 and 4, namely that the centre of mass of a body is the point through which its total weight acts. To avoid algebraic

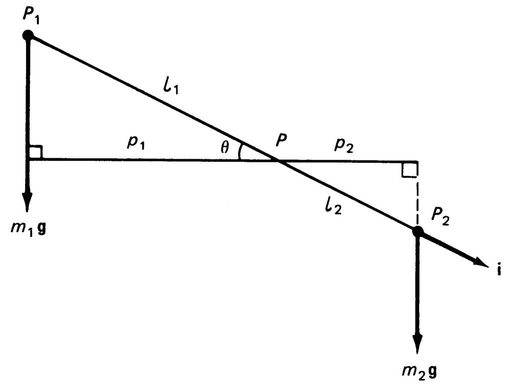


Fig. 5.4

complexity we only consider the case of two particles P_1 and P_2 of weights $m_1\mathbf{g}$ and $m_2\mathbf{g}$ as shown in Fig. 5.4. However the proof which follows can be extended without difficulty to the case of n particles, and to a continuous distribution of mass. Let $\mathbf{i}$ be a unit vector parallel to $\overrightarrow{P_1P_2}$ and let us measure position vectors relative to P_1. Thus $\mathbf{r}_1 = \mathbf{0}$, $\mathbf{r}_2 = (l_1 + l_2)\mathbf{i}$ and, by (3.1), the position vector $\bar{\mathbf{r}}$ of the centre of mass P is given by

$$\bar{\mathbf{r}} = \frac{m_2}{(m_1 + m_2)}(l_1 + l_2)\mathbf{i}.$$

However, from Fig. 5.4, $\bar{\mathbf{r}} = l_1\mathbf{i}$. Therefore

$$l_1 = \frac{m_2 l_1 + m_2 l_2}{m_1 + m_2} \quad \Rightarrow \quad m_1 l_1 = m_2 l_2. \tag{5.5}$$

By the preceding work in this section, the weight $m_1\mathbf{g}$ acting through P_1 can be replaced by a force $m_1\mathbf{g}$ acting through P together with a couple of moment L_1, where $L_1 = m_1 g p_1$ (since $m_1\mathbf{g}$ has an anti-clockwise moment about P). Similarly the weight $m_2\mathbf{g}$ acting through P_2 can be replaced by a force $m_2\mathbf{g}$ acting through P together with a couple of moment L_2, where $L_2 = -m_2 g p_2$. Therefore the system of two separate weights is equivalent to a force $(m_1 + m_2)\mathbf{g}$ acting through P, together with a couple of moment L, where, from the diagram and using (5.5),

$$\begin{aligned} L = L_1 + L_2 &= (m_1 p_1 - m_2 p_2)g \\ &= (m_1 l_1 - m_2 l_2)g\cos\theta = 0. \end{aligned}$$

Since $L = 0$, the system of two separate weights is equivalent to a single force $(m_1 + m_2)\mathbf{g}$, the total weight, acting through the centre of mass.

This illustration of the theory raises the central question: 'Can an *arbitrary* system of coplanar forces be replaced by a single force?'

The answer to this question depends on whether $\mathbf{F}$ in (5.3) is zero or non-zero. We call $\mathbf{F}$ the *resultant of the forces*—sometimes *resultant force* or simply *resultant*. When $\mathbf{F} \neq \mathbf{0}$ the answer to our question is 'yes': a system of coplanar forces with non-zero resultant can be replaced by a single force. More precisely, a system of coplanar forces with $\mathbf{F} \neq \mathbf{0}$ can be replaced by a single force $\mathbf{F}$ with a definite line of action—the *line of action of the resultant*. This conclusion is illustrated by the system of two particles in Fig. 5.4, for we have shown that the original system of two forces, one the weight $m_1\mathbf{g}$ with line of action vertically through P_1 and the other the weight $m_2\mathbf{g}$ with line of action vertically through P_2, can be replaced by one force $(m_1 + m_2)\mathbf{g}$ with line of action vertically through P.

On the other hand, when $\mathbf{F} = \mathbf{0}$, so that the resultant of the forces is zero, the answer to our question is 'no' (except in one special circumstance). A set of coplanar forces whose resultant is zero cannot, in general, be replaced by a single force; as we shall see, such a set of forces can be replaced by a couple. In order to establish these statements we need a further result, which forms part of the next example.

Example 2 With respect to the basis $\{\mathbf{i}, \mathbf{j}\}$, a force $\mathbf{F}_1$ has components (X_1, Y_1), so that $\mathbf{F}_1 = X_1\mathbf{i} + Y_1\mathbf{j}$, and acts at the point P_1 with position vector $x_1\mathbf{i} + y_1\mathbf{j}$. Show that the moment L_1 of $\mathbf{F}_1$ about the point P with position vector $x_P\mathbf{i} + y_P\mathbf{j}$ is given by

$$L_1 = (x_1 - x_P)Y_1 - (y_1 - y_P)X_1. \qquad (5.6)$$

Show that L_1 is unchanged if $\mathbf{F}_1$ acts at any other point of its line of action.

The situation is illustrated in Fig. 5.5. The moment L_1 of $\mathbf{F}_1$ about P is equal

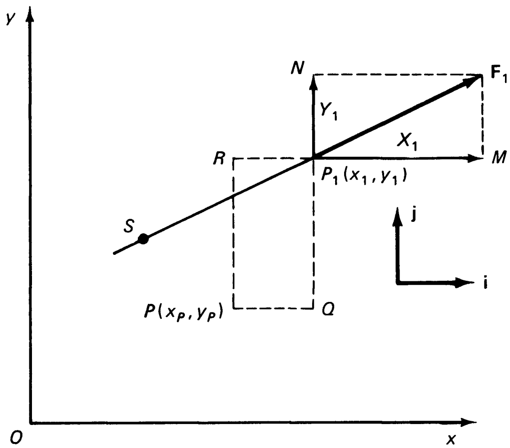


Fig. 5.5

to the sum of the moments about P of $X_1\mathbf{i}$ and $Y_1\mathbf{j}$. Hence, with due regard to our convention that anti-clockwise moments are positive, we obtain

$$L_1 = Y_1 \,.\, QP - X_1 \,.\, RP.$$

Since $QP = (x_1 - x_P)$ and $RP = (y_1 - y_P)$, we obtain (5.6). The reader may verify that (5.6) also gives the correct moment when some (or all) of X_1, Y_1, $(x_1 - x_P)$, $(y_1 - y_P)$ are negative.

Let S, with position vector $x_S\mathbf{i} + y_S\mathbf{j}$, be any other point on the line of action of $\mathbf{F}_1$, and let us suppose that $\mathbf{F}_1$ acts at S instead of P_1. Then $\overrightarrow{SP_1}$ is parallel to $\mathbf{F}_1$, so that

$$\overrightarrow{SP_1} = \lambda\mathbf{F}_1 = \lambda(X_1\mathbf{i} + Y_1\mathbf{j})$$

for some scalar λ. But $\overrightarrow{SP_1} = \overrightarrow{OP_1} - \overrightarrow{OS}$. Hence

$$(x_1\mathbf{i} + y_1\mathbf{j}) - (x_S\mathbf{i} + y_S\mathbf{j}) = \lambda(X_1\mathbf{i} + Y_1\mathbf{j})$$

$$\Rightarrow \quad x_S = x_1 - \lambda X_1, \quad y_S = y_1 - \lambda Y_1.$$

The moment L_1' of $\mathbf{F}_1$ about P is obtained from (5.6) by replacing (x_1, y_1) by (x_S, y_S). Hence

$$L_1' = (x_S - x_P)Y_1 - (y_S - y_P)X_1 = (x_1 - \lambda X_1 - x_P)Y_1 - (y_1 - \lambda Y_1 - y_P)X_1$$

$$\Rightarrow \quad L_1' = (x_1 - x_P)Y_1 - (y_1 - y_P)X_1 = L_1,$$

as required.

The last part of this example provides the justification for an assumption that we have made implicitly several times, namely that the effect of a force (acting on a rigid body) depends only on its line of action, and not on the particular point on the line at which the force acts.

5.3 The reduction of a set of plane forces

We continue to consider a rigid body acted on by n coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ with lines of action $l_1, l_2, \ldots, l_n$. We shall suppose, as in Example 2, that the plane in which the forces act is the xy-plane, and that the components of $\mathbf{F}_r$ with respect to the basis $\{\mathbf{i}, \mathbf{j}\}$ are X_r, Y_r. Hence, for $r = 1, 2, \ldots, n$,

$$\mathbf{F}_r = X_r\mathbf{i} + Y_r\mathbf{j}. \tag{5.7}$$

The resultant $\mathbf{F}$ of this set of forces is defined in (5.3). Therefore

$$\mathbf{F} = (X_1\mathbf{i} + Y_1\mathbf{j}) + (X_2\mathbf{i} + Y_2\mathbf{j}) + \ldots + (X_n\mathbf{i} + Y_n\mathbf{j})$$

$$\Rightarrow \quad \mathbf{F} = (X_1 + X_2 + \ldots + X_n)\mathbf{i} + (Y_1 + Y_2 + \ldots + Y_n)\mathbf{j}$$

$$\Rightarrow \quad \mathbf{F} = X\mathbf{i} + Y\mathbf{j} \text{ where } X = \Sigma X_r,\ Y = \Sigma Y_r. \tag{5.8}$$

The summation in (5.8) indicated by capital Greek sigma is over all values of r from $r = 1$ to $r = n$.

From our work in §5.2 we know that this system of forces can be replaced by a force **F** acting through an arbitrary point P together with a couple of moment L, where, according to (5.4), L is the sum of the moments of the separate forces about P. As in Example 2, suppose P has position vector $x_P\mathbf{i} + y_P\mathbf{j}$ and let P_r, with position vector $x_r\mathbf{i} + y_r\mathbf{j}$, be any point on l_r the line of action of $\mathbf{F}_r$. From (5.6), the moment L_r of $\mathbf{F}_r$ about P satisfies

$$L_r = (x_r - x_P)Y_r - (y_r - y_P)X_r = (x_rY_r - y_rX_r) - x_PY_r + y_PX_r.$$

Hence, from (5.4), the total moment L satisfies

$$\begin{aligned} L &= L_1 + L_2 + \ldots + L_n, \\ &= [(x_1Y_1 - y_1X_1) - x_PY_1 + y_PX_1] + [(x_2Y_2 - y_2X_2) - x_PY_2 + y_PX_2] \\ &\quad + \ldots + [(x_nY_n - y_nX_n) - x_PY_n + y_PX_n]. \end{aligned}$$

Hence, using capital sigma to denote summation over all values r from 1 to n,

$$L = \Sigma(x_rY_r - y_rX_r) - x_P\Sigma Y_r + y_P\Sigma X_r. \tag{5.9}$$

In obtaining (5.9) we have used the fact that, having chosen P, the values of x_P and y_P are the same for all particles and can therefore be taken outside the summation sign.

We now return to the question raised in §5.2. Can such a system of forces be replaced by a single force? We have seen that for an arbitrary point P the system can be replaced by a single force **F** acting at P, where **F** is given by (5.8), together with a couple of moment L, where L is given by (5.9). The answer to the question is therefore 'yes' if, and only if, $L = 0$. From (5.8) and (5.9) we find

$$L = 0 \quad \Leftrightarrow \quad x_PY - y_PX = L_O \tag{5.10}$$

where $L_O = \Sigma(x_rY_r - y_rX_r)$. The symbol L_O for $\Sigma(x_rY_r - y_rX_r)$ is an appropriate one because it follows from (5.6) that L_O is the total moment of the forces about the origin O.

There are two possible cases: (i) $\mathbf{F} \neq \mathbf{0}$ and (ii) $\mathbf{F} = \mathbf{0}$.

(i) $\mathbf{F} \neq \mathbf{0}$. In this case, since $\mathbf{F} = X\mathbf{i} + Y\mathbf{j}$, not both of X and Y are zero. Hence (5.10) is a linear equation relating x_P and y_P, or, in other words, the point P must lie on the *straight line* l with equation

$$xY - yX = L_O. \tag{5.11}$$

This straight line l is the line of action of the resultant and is parallel to **F**. The system can be replaced by a single force **F** acting at any point of l.

(ii) $\mathbf{F} = \mathbf{0}$. In this case both X and Y are zero. The condition (5.10) then reduces to $L = 0 \Leftrightarrow L_O = 0$, where $L_O = \Sigma(x_rY_r - y_rX_r)$ is independent of P, since it is the total moment of all the forces about the fixed point O. In general L_O will not be zero, so L will not be zero either. The system of forces cannot be replaced by a single force in this case. However, since X and Y are now both

zero, it follows from (5.9) that $L = L_O$ for all choices of the point P; therefore the system of forces is equivalent to a couple of moment L_O.

Finally we consider the special case for which $\mathbf{F} = \mathbf{0}$ and $L_O = 0$. In this case the system of forces has no effect whatsoever. The resultant force is zero and the forces have zero moment about every point. Hence we have, in particular, justified the remarks following (4.15).

Example 3 $ABCD$ is a square of side 2 m. Six forces of magnitude 1 N, 3 N, 4 N, 2 N, p N and q N act along AB, BC, CD, DA, AC and DB respectively. Find the values of p and q if (i) the system reduces to a couple, and (ii) the system reduces to a single force acting through A and P, the mid-point of BC. In case (i) find the magnitude of the couple, and in case (ii) find the magnitude of the force.

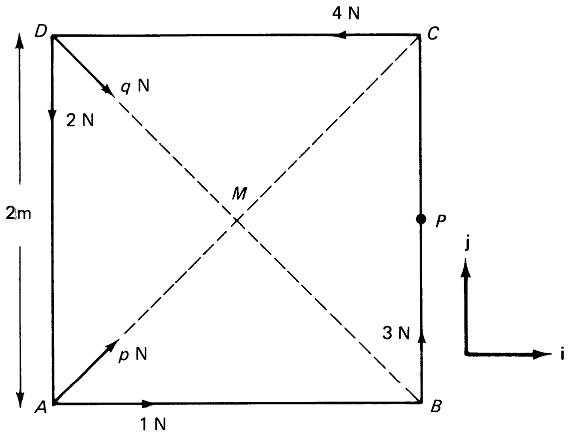


Fig. 5.6

The situation is illustrated in Fig. 5.6 and unit vectors $\mathbf{i}$ and $\mathbf{j}$ are taken as shown. Let $(X\mathbf{i} + Y\mathbf{j}) = \mathbf{F}$ be the resultant force. From (5.7) and (5.8),

$$X = (1 - 4 + p\cos 45^\circ + q\cos 45^\circ)\ \mathrm{N} = [\tfrac{1}{2}(p+q)\sqrt{2} - 3]\ \mathrm{N}, \quad (5.12)$$

$$Y = (3 - 2 + p\cos 45^\circ - q\cos 45^\circ)\ \mathrm{N} = [\tfrac{1}{2}(p-q)\sqrt{2} + 1]\ \mathrm{N}. \quad (5.13)$$

Let L_A be the moment of the forces about A. Since $AM = \sqrt{2}$ m,

$$L_A = (3 \times 2 + 4 \times 2 - q \times \sqrt{2})\ \mathrm{Nm} = (14 - q\sqrt{2})\ \mathrm{Nm}. \quad (5.14)$$

(i) The system reduces to a couple if $\mathbf{F} = \mathbf{0}$, that is if $X = Y = 0$.

$$X = Y = 0 \quad \Rightarrow \quad p + q = 3\sqrt{2}, \quad p - q = -\sqrt{2}$$

$$\Rightarrow \quad p = \sqrt{2}, \quad q = 2\sqrt{2}.$$

For these values of p and q, $L_A = 10$ Nm and this is the moment of the couple to which the system reduces.

(ii) We know that the points A and P lie on the line of action of the resultant. If we take the point A as the origin, we can use (5.11) with L_A given by (5.14) replacing L_O.

A lies on the line with equation (5.11) $\Rightarrow$ $0 . Y - 0 . X = L_A$.

P lies on the line with equation (5.11) $\Rightarrow$ $2 . Y - 1 . X = L_A$.

Hence, from (5.12), (5.13) and (5.14),

$$14 - q\sqrt{2} = 0 \quad \Rightarrow \quad q = 7\sqrt{2}, X = 2Y \quad \Rightarrow \quad p = 16\sqrt{2}.$$

For these values of p and q, $X = 20$ N, $Y = 10$ N. Hence

$$|\mathbf{F}| = \sqrt{(X^2 + Y^2)} = 10\sqrt{5}\ \text{N}.$$

Before considering another example, we pause to make two remarks. In (5.14) we have used the unit Nm ('newton metre') for moment, whereas in Chapter 4 we used the unit kg m^{2}s^{-2}. However, since 1 N = 1 kg ms^{-2}, the units are the same.

The second point is to note that there is an alternative way of doing Example 3 (ii) which does not make explicit use of (5.11). Since the points A and P lie on the line of action of the resultant, the forces have no moment about A or P. Thus $L_A = 0$ (as before), and taking moments about P directly, we find, since P is a distance $\frac{1}{2}\sqrt{2}$ m from both AC and BD,

$$0 = (1 \times 1) + (4 \times 1) + (2 \times 2) + q(\tfrac{1}{2}\sqrt{2}) - p(\tfrac{1}{2}\sqrt{2})$$

$$\Rightarrow \quad (p - q)\sqrt{2} = 18.$$

Since $L_A = 0$, $q = 7\sqrt{2}$, and the condition for the moment about P to be zero gives $p = 16\sqrt{2}$ as before.

Example 4 A system of coplanar forces has total moments N, pN and qN about the points $(0, 0)$, $(a, 0)$ and $(0, b)$ respectively. This system is equivalent to a single force with components (X, Y) acting through the point $(x_1, 0)$. Determine X and Y in terms of a, b, p, q and N. Show that an equation of the line of action of the resultant is

$$(1 - p)\frac{x}{a} + (1 - q)\frac{y}{b} = 1,$$

and hence find x_1.

We use (5.9) in the form $L = L_O - x_P Y + y_P X$.

$(x_P, y_P) = (0, 0) \quad \Rightarrow \quad L = N = L_O$.

$(x_P, y_P) = (a, 0) \quad \Rightarrow \quad L = pN = N - aY \quad \Rightarrow \quad Y = \dfrac{N}{a}(1 - p)$.

$(x_P, y_P) = (0, b) \quad \Rightarrow \quad L = qN = N + bX \quad \Rightarrow \quad X = -\dfrac{N}{b}(1 - q)$.

The line of action of the resultant is given by (5.11) with $L_O = N$ and X, Y as above. Thus the line of action of the resultant is

$$(1-p)\frac{x}{a} + (1-q)\frac{y}{b} = 1.$$

Since we are given that this line passes through $(x_1, 0)$, we find

$$(1-p)\frac{x_1}{a} = 1 \quad \Rightarrow \quad x_1 = \frac{a}{(1-p)}.$$

5.4 Sets of coplanar forces acting on rigid bodies

In mechanics the term *rigid body* means a collection of particles rigidly joined together. No real body is truly rigid but the concept is a very useful mathematical model since experiments show that many bodies move in close agreement with the results of calculations based on the assumption that they are rigid.

Let us suppose that the rigid body is composed of n particles $P_1, P_2, \ldots, P_n$. Consider any one of these particles, say P_r. The forces acting on P_r will be of two types.

(i) First, P_r will be acted on by *external forces*, which are those forces due to agencies exterior to the body. The obvious example of an external force acting on P_r is its weight due to the Earth's gravitational attraction. But there are other types of external force. Consider, for example, the beam shown in Fig. 5.2. The particle of the beam in contact with the support at P experiences, as an external force, the force exerted by the support at P on the beam. Let $\mathbf{F}_r$ denote the total external force exerted on P_r.

(ii) All other forces acting on P_r are due to the presence of the other particles of which the body is composed. We shall call such forces *internal forces*, and they act (in response partly to the external forces) in such a way that the body is constrained to remain rigid. For this reason the internal forces acting in a rigid body are often called *forces of constraint*. Although we do not need to enquire into the physical mechanisms causing these forces, we shall, to cover all possibilities, suppose that each particle exerts a force of constraint on every other particle. Let $\mathbf{F}_{rs}$ be the force exerted on the particle P_r by the particle P_s. Naturally $\mathbf{F}_{rr} = \mathbf{0}$ since a particle cannot exert a force of constraint on itself! By Newton's Third Law

$$\mathbf{F}_{rs} = -\mathbf{F}_{sr}. \tag{5.15}$$

Furthermore $\mathbf{F}_{rs}$ and $\mathbf{F}_{sr}$ have the same line of action, namely the line joining P_r to P_s.

If the rigid body is in equilibrium, then every individual particle within it must also be in equilibrium. By (4.1) the particle P_r can be in equilibrium only if the sum of all the forces acting on it is zero, that is only if the sum of $\mathbf{F}_r$ and all the internal forces $\mathbf{F}_{rs}$ is zero. By adding together the corresponding results for every particle, we conclude that, for the body to be in equilibrium, the sum of all the external forces $\mathbf{F}_r$ and all the internal forces acting on all the

particles is zero. But by (5.15), the sum of all the internal forces acting on all the particles is itself zero, since the terms in the sum cancel out pair by pair. Thus, for equilibrium, the sum $\mathbf{F}$ of all the external forces acting on the body is zero, that is

$$\mathbf{F} = \mathbf{0}, \quad \text{where } \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n. \tag{5.16}$$

This provides the justification that the first part of (4.15) is *necessary* for the rigid body to be in equilibrium.

Since the sum of all the forces acting on P_r is zero, the total moment of all these forces about any point is zero. By adding together the corresponding results for every particle, we conclude that, for the body to be in equilibrium, the sum of the moments about any point of all the $\mathbf{F}_r$ and all the internal forces acting on all the particles is zero. But, by (5.15) and the fact that $\mathbf{F}_{rs}$ and $\mathbf{F}_{sr}$ have the same line of action, the sum of the moments about any point of all the internal forces acting on all the particles is itself zero, since the terms in the sum cancel out pair by pair. Thus, for equilibrium, the sum L of the moments about any point of all the external forces is zero. Let L_r be the moment about any point of $\mathbf{F}_r$. Then, for equilibrium,

$$L = 0, \quad \text{where } L = L_1 + L_2 + \ldots + L_n. \tag{5.17}$$

This provides the justification that the second part of (4.15) is *necessary* for the rigid body to be in equilibrium.

As we have not yet used the fact that the body is rigid, we see that (5.16) and (5.17) are necessary conditions for the equilibrium of *any* system of particles. The distinguishing feature of the system of particles forming a rigid body is that (5.16) and (5.17) are also *sufficient* conditions for the rigid body to remain in equilibrium.

The essential property of a rigid body that enables this statement to be proved in the next paragraph is that, if one point of the body is fixed, then the only way in which it can move is to turn about an axis through this fixed point.

We begin by showing that, when $\mathbf{F} = \mathbf{0}$, the centre of mass of the body remains at rest provided that it is initially at rest. We let the particles P_1, P_2, $\ldots$, P_n have masses $m_1, m_2, \ldots, m_n$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. By (3.3), the position vector $\bar{\mathbf{r}}$ of the centre of mass of the body satisfies

$$M\bar{\mathbf{r}} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n,$$

where M, defined by

$$M = m_1 + m_2 + \ldots + m_n, \tag{5.18}$$

is the total mass of the body. Hence the acceleration $\ddot{\bar{\mathbf{r}}}$ of the centre of mass satisfies

$$M\ddot{\bar{\mathbf{r}}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + \ldots + m_n\ddot{\mathbf{r}}_n. \tag{5.19}$$

But by Newton's Second Law, $m_1\ddot{\mathbf{r}}_1$ is equal to the sum of all the forces acting on P_1, and similarly for all particles. Hence the right-hand side of (5.19) is equal

to the sum of all the forces, external and internal, acting on all the particles. Since, as we have just shown, the sum of the internal forces is zero, the right-hand side of (5.19) is equal to the sum of the external forces, namely **F** defined in (5.16). Therefore (5.19) becomes

$$M\ddot{\mathbf{r}} = \mathbf{F}. \tag{5.20}$$

Equation (5.20) is an important equation for the development of mechanics at a more advanced level than that in this book. Here we consider only the case of direct relevance to this chapter, namely $\mathbf{F} = \mathbf{0}$. Then, from (5.20), $\ddot{\mathbf{r}} = \mathbf{0}$ so that $\dot{\mathbf{r}}$, the velocity of the centre of mass, is constant. Hence the centre of mass is at rest provided that it is initially at rest. Thus (5.16) is a *sufficient* condition for one point of the body, namely its centre of mass, to be at rest.

Since we are considering only cases in which the rigid body is acted on by forces all lying in a plane, the only way it can move when one point is fixed is to turn about an axis through this point, the axis being perpendicular to the plane in which the forces act. It is obvious that all points on the axis are also at rest. We shall show that, when $L = 0$, where L is the total moment of the external forces about the axis, the body is not turning, if it is not turning initially.

Let P be a particle of mass m, and let O be the point on the axis in the plane through P perpendicular to the axis. Since the body is rigid and turning about the axis through O, the particle P moves in a circle of centre O and radius OP. We denote OP by r, and we let OP make an angle θ with a fixed direction OX as shown in Fig. 5.7. Note that we choose to measure θ so that it increases as OP rotates in an anti-clockwise sense. From our knowledge of circular motion (as covered for example in Examples 7 and 8 in Chapter 2 of *Newton's Laws and Particle Motion*), we know that the component of the acceleration **a** in the direction of $\hat{\boldsymbol{\theta}}$, the unit vector perpendicular to OP, is $r\ddot{\theta}$, where $\ddot{\theta}$ is the *angular acceleration* (the derivative with respect to time of $\dot{\theta}$, the angular speed with which the body is turning). Hence, by Newton's Second Law,

$$mr\ddot{\theta} = F_\theta,$$

where F_θ is the component in the direction of $\hat{\boldsymbol{\theta}}$ of the total force **F** acting on P. The total moment N of **F** about the axis is obviously rF_θ. Hence

$$mr^2\ddot{\theta} = N. \tag{5.21}$$

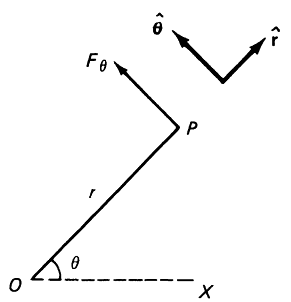


Fig. 5.7

An equation like (5.21) holds for all particles and, since the body is rigid, the value of $\ddot{\theta}$ (but not of m, r or N) for all particles is the same. When we sum for all these equations, we obtain

$$I_O\ddot{\theta} = L \quad \text{where } I_O = \Sigma mr^2 \text{ and } L = \Sigma N. \tag{5.22}$$

We know that the total moment of the internal forces about the axis is itself zero, so that L is the total moment of the external forces about the axis, as defined in (5.17). From its definition, I_O is independent of the way in which the body is moving and of the forces acting. Hence I_O is a property of the body's geometry alone and is known as the *moment of inertia about the axis through 0*, or simply the *moment of inertia* when the position of the axis is clear.

We shall make use of (5.22) in Chapter 6. Here we consider only the case when $L = 0$. Then, from (5.22), $\ddot{\theta} = 0$ so that $\dot{\theta}$, the angular speed of the body, is constant. Hence the body does not start to turn if initially it is not turning. Thus (5.17) is a sufficient condition for a rigid body with one point fixed to be at rest when acted on by a set of coplanar forces.

Exercise 5

1 Forces of magnitude 4 N, 4 N, 6 N act respectively along AB, BC, CA, where ABC is an equilateral triangle of side 1 m. The line of action of the resultant of these forces meets BC produced at D. Calculate the magnitude of the resultant and the distance BD.

2 A system of forces acts in the plane of an equilateral triangle PQR of side a. This system has moments K, K, 0 about P, Q, R respectively. Show that the resultant of this system is parallel to QP and find its magnitude.

3 In a rectangle $ABCD$, $AB = 4$ m and $AD = BC = 3$ m. Forces of magnitude 3 N, 5 N, 4 N, 6 N, p N and q N act along AB, BC, CD, DA, BD and AC respectively. Show that this system of forces cannot be in equilibrium.

(a) If the system reduces to a couple, show that $q = 7p$.

(b) If the system reduces to a single resultant force acting through B, show that $q = 15$. Given also that $p = 10$, find the magnitude of the resultant.

4 A non-uniform, heavy beam $ABCD$ rests horizontally on supports at B and C, where $AB = BC = CD = a$. When a load of mass M is hung from A, the beam is on the point of rotating about B. When now an additional load of mass $5M$ is hung from D, the beam is on the point of rotating about C. Find the mass of the beam, and show that its centre of mass is at a distance $a/4$ from B.

5 Masses M, $5M$, αM, $2M$, βM, $3M$ are attached to points A, B, C, D, E, F, respectively, equally spaced around the circumference of a uniform circular disc of radius a. The disc is free to rotate in a vertical plane about a fixed horizontal axis through its centre O. When E is vertically below O the disc is kept at rest by a couple of moment $\frac{1}{2}Mga\sqrt{3}$. When D is vertically below O the disc is kept at rest by a couple of moment $-\frac{1}{2}Mga\sqrt{3}$. Find α and β.

6 A uniform cylinder of radius a and mass M rests with its curved surface in contact with two fixed rough planes, each inclined at an angle β to the horizontal. The planes intersect in a line parallel to the axis of the cylinder and perpendicular to the line of greatest slope of each plane. The coefficient of friction between each plane and the cylinder is μ. A couple of moment L is applied to the cylinder in a plane perpendicular to its axis. Find L given that the cylinder is on the point of slipping.

6 The motion of a rigid body about a fixed axis

6.1 Introduction

The purpose of this final chapter is to investigate the motion of a rigid body turning about a fixed axis. We shall suppose that the axis is smooth, so that its bearings do not exert any frictional couple and therefore do not retard the rotation of the body about the axis. As in Chapters 4 and 5 we shall consider only situations in which the external forces act in a plane perpendicular to the axis.

The basic situation is illustrated in Fig. 6.1. The angle θ is the angle between a line ON fixed in space and a line OA fixed in the body, with θ measured in such a way that it increases as the body rotates about O in an anti-clockwise sense. In Fig. 6.1 the axis is, of course, normal to the page. From equation (5.22), we know that

$$I_O\ddot{\theta} = L, \tag{6.1}$$

where L is the moment of the external forces about the axis through O and I_O is the moment of inertia about the axis through O. From (5.22), I_O is defined by

$$I_O = \Sigma mr^2, \tag{6.2}$$

where the summation is over all particles of the body. For a particle of the body of mass m, the length r is its distance from the axis and so is the radius of the circle in which the particle moves.

Before applying (6.1) to some specific problems, we use integration to find the values of I_O for some simple bodies.

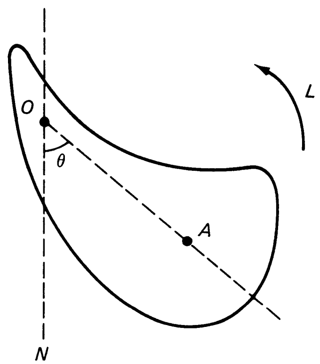


Fig. 6.1

6.2 The determination of moments of inertia

The method of determining moments of inertia is very similar in principle to that employed in §3.4 for finding centres of mass for continuous distributions of mass. We begin with a simple, but important, example.

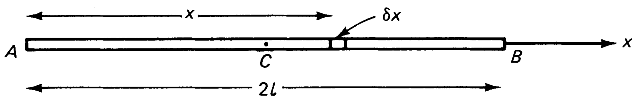


Fig. 6.2

Example 1 Find the moments of inertia of a thin, uniform, rigid rod of mass M and length $2l$ about axes perpendicular to the rod through (i) one end of the rod, and (ii) the mid-point of the rod.

As shown in Fig. 6.2, we denote the rod by AB and measure distance x along the rod from A. The small element of length δx shown has mass $M\delta x/2l$.

(i) The distance of the element from the axis through A is x. By (6.2), this element contributes an amount δI_A to I_A, the moment of inertia through A, where

$$\delta I_A = \frac{M\delta x}{2l} x^2.$$

We now obtain I_A by integrating from $x = 0$ to $x = 2l$, so that

$$I_A = \left(\frac{M}{2l}\right) \int_0^{2l} x^2 \, \mathrm{d}x = \frac{4}{3} Ml^2. \tag{6.3}$$

(ii) The distance of the element from an axis through the mid-point C is $(x - l)$, so that

$$I_C = \left(\frac{M}{2l}\right) \int_0^{2l} (x - l)^2 \, \mathrm{d}x = \left(\frac{M}{6l}\right) \Big[(x - l)^3\Big]_0^{2l} = \frac{1}{3} Ml^2. \tag{6.4}$$

The results of Example 1 illustrate a very useful result known as the *theorem of parallel axes*. The theorem relates the moment of inertia I_O about an axis through an arbitrary point O of the body to the moment of inertia I_G about a *parallel* axis through the centre of mass G of the body. Let us consider a typical particle P of mass m and, as shown in Fig. 6.3, let the plane perpendicular to the axes through P meet the axes through O and G in the points O' and G' respectively. Take axes centred at G', with the x-axis through O'. We let $d = O'G'$ be the distance between the parallel axes. The contribution to I_G from P is

$$m(G'P)^2 = m(x^2 + y^2),$$

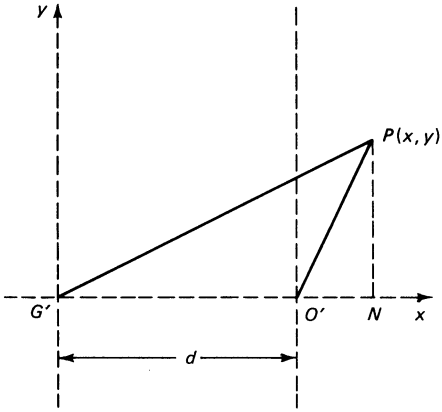


Fig. 6.3

where (x, y) are the coordinates of P. Hence, from the definition in (6.2),

$$I_G = \Sigma m(x^2 + y^2). \tag{6.5}$$

On the other hand, the contribution of P to I_O is

$$m(O'P)^2 = m[(O'N)^2 + (NP)^2] = m[(x-d)^2 + y^2]$$
$$= m(x^2 + y^2 - 2dx + d^2),$$

since $O'N = G'N - G'O' = (x - d)$. Hence, from the definition in (6.2),

$$I_O = \Sigma m(x^2 + y^2 - 2dx + d^2) = I_G - 2d\Sigma mx + d^2\Sigma m, \tag{6.6}$$

using (6.5). In (6.6) the last term can be written Md^2, where

$$M = \Sigma m \tag{6.7}$$

is the total mass of the body, as in equation (5.18). We now consider the term $2d\Sigma mx$ on the right-hand side of (6.6). Since G is the centre of mass of the body, the x component of $\bar{\mathbf{r}}$, the position vector of G with respect to G', is zero. Hence, taking the x component of (3.3), we see that $\Sigma mx = 0$. Therefore

$$I_O = I_G + Md^2, \tag{6.8}$$

and this is the mathematical statement of the theorem.

In Example 1 (ii) we obtained the result $I_C = \frac{1}{3}Ml^2$. Since C is the centre of mass of the rod, and since $AC = l$, we obtain from (6.8)

$$I_A = \tfrac{1}{3}Ml^2 + Ml^2 = \tfrac{4}{3}Ml^2,$$

which is in agreement with the direct calculation in Example 1(i).

Equation (6.8) shows that the moment of inertia of a body about an axis in a given direction is *least* when the axis passes through the centre of mass.

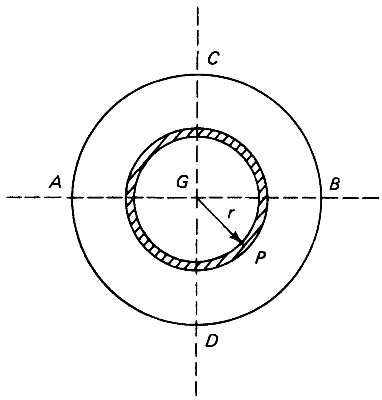


Fig. 6.4

Example 2 Find the moment of inertia of a uniform circular disc of mass M and radius a about an axis perpendicular to the disc and passing through its centre G. Use the theorem of parallel axes to find the moment of inertia of the disc about an axis perpendicular to the plane of the disc through a point on its edge.

The situation is illustrated in Fig. 6.4. Consider a ring of radius r and thickness δr. Its area is $2\pi r\delta r$ so that its mass is $(2\pi r\delta r/\pi a^2)M = 2r\delta rM/a^2$. Each element of the ring is at an equal distance r from the axis through G. Hence the contribution of the ring to I_G is $(2r\delta rM/a^2)r^2 = (2M/a^2)r^3\delta r$. By integration

$$I_G = \frac{2M}{a^2}\int_0^a r^3\,\mathrm{d}r = \tfrac{1}{2}Ma^2. \tag{6.9}$$

Since G is the centre of mass, and since $GA = a$, where A is any point on the edge of the disc, it follows from (6.8) and (6.9) that

$$I_A = \tfrac{1}{2}Ma^2 + Ma^2 = \tfrac{3}{2}Ma^2.$$

The theorem of parallel axes (6.8) applies to all bodies. For the special case when the body is a *lamina* (but only in this case), there is also another theorem, the *theorem of perpendicular axes*, which is sometimes useful. This theorem relates the moment of inertia I_O about an axis perpendicular to the lamina through a point O to the moments of inertia I_x and I_y about any two perpendicular axes through O, both lying in the plane of the lamina. The situation is illustrated in Fig. 6.5, where I_x and I_y are the moments of inertia about Ox and Oy, and I_O is the moment of inertia about the axis through O perpendicular to the lamina (which is, of course, the z-axis Oz in this case). The contributions δI_x, δI_y and δI_O to I_x, I_y and I_O, respectively, of a typical particle P of mass m are

$$\delta I_x = m(PM)^2, \quad \delta I_y = m(PN)^2, \quad \delta I_O = m(PO)^2.$$

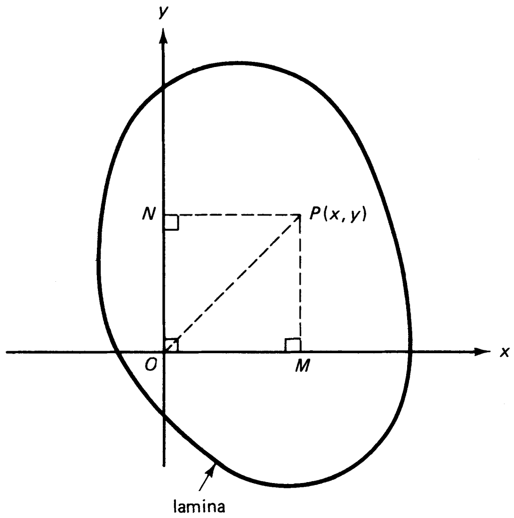


Fig. 6.5

$$\text{Pythagoras' Theorem} \quad \Rightarrow \quad (PM)^2 + (PN)^2 = (PO)^2$$
$$\Rightarrow \quad \delta I_x \;+\; \delta I_y \;=\; \delta I_O.$$

Hence, on summing over all particles, we obtain

$$I_x + I_y = I_O, \tag{6.10}$$

which is the mathematical statement of the theorem of perpendicular axes.

The circular disc in Example 2 lends itself to a typical illustration of the theorem of perpendicular axes. As shown in Fig. 6.4, let AB and CD be two perpendicular diameters of the disc. By symmetry, the moment of inertia about these two diameters is the same; let its value be I_1. From equation (6.10), $2I_1 = I_G = \frac{1}{2}Ma^2$, using the result of Example 2. Hence $I_1 = \frac{1}{4}Ma^2$.

The next example illustrates the use of both the theorem of parallel axes, equation (6.8), and the theorem of perpendicular axes, equation (6.10).

Example 3 The mass of a rectangular lamina $ABCD$ is M, and $AB = CD = l$, $AD = BC = b$. Find the moment of inertia of the lamina about

(i) the side AD,
(ii) an axis parallel to AD through the centre O of the rectangle,
(iii) an axis through O perpendicular to the plane of the rectangle,
(iv) an axis through A perpendicular to the plane of the rectangle.

Consider a thin strip of thickness δx whose centre G is a distance x from O. The rectangle and the strip are shown in Fig. 6.6. The mass of the strip is $M\delta x/b$.

(i) From Example 1(i), the moment of inertia δI_{AD} of the strip about AD satisfies

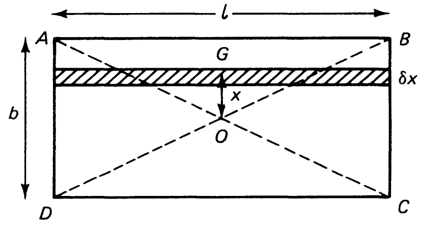


Fig. 6.6

$$\delta I_{AD} = \frac{4}{3}\left(\frac{M\delta x}{b}\right)\left(\frac{l}{2}\right)^2 = \frac{1}{3}(Ml^2/b)\delta x.$$

(Note that the length of the rod in Example 1 is $2l$, whereas the strip in the present example has length l.) Hence

$$I_{AD} = \frac{1}{3}\left(\frac{Ml^2}{b}\right)\int_{-b/2}^{b/2} \mathrm{d}x = \frac{1}{3}Ml^2.$$

(ii) By the theorem of parallel axes, the moment of inertia of the lamina about an axis through O parallel to AD is I'_{AD}, where (because O is the centre of mass of the lamina)

$$I_{AD} = I'_{AD} + M\left(\frac{l}{2}\right)^2 \quad \Rightarrow \quad I'_{AD} = \frac{1}{12}Ml^2.$$

(iii) By symmetry, the moment of inertia I'_{AB} about an axis through O parallel to AB is given by $I'_{AB} = (1/12)Mb^2$. By the theorem of perpendicular axes, the moment of inertia I_O about an axis through O perpendicular to the plane of the rectangle satisfies

$$I_O = I'_{AD} + I'_{AB} = \tfrac{1}{12}M(b^2 + l^2). \tag{6.11}$$

(iv) We can find I_A, the moment of inertia about an axis through A perpendicular to the plane of the rectangle, in either of two ways.

(a) $\qquad (6.8) \quad \Rightarrow \quad I_A = I_O + M(OA)^2$

$$= \frac{1}{12}M(b^2 + l^2) + M\left[\left(\frac{b}{2}\right)^2 + \left(\frac{l}{2}\right)^2\right] \tag{6.12}$$

$$\Rightarrow \quad I_A = \tfrac{1}{3}M(b^2 + l^2).$$

(b) By symmetry, using the result of (i), the moment of inertia I_{AB} about AB is given by $I_{AB} = \frac{1}{3}Mb^2$. Then

$$(6.10) \quad \Rightarrow \quad I_A = I_{AB} + I_{AD} = \tfrac{1}{3}M(b^2 + l^2) \quad \text{as in (a).}$$

The methods developed in this section can be applied to determine moments of inertia of a rigid body which is not a lamina. As a first example consider

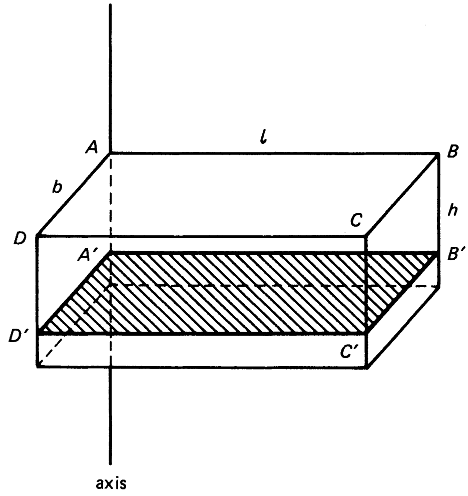


Fig. 6.7

the uniform cuboid shown in Fig. 6.7. Its base is a rectangle equal to that considered in Example 3 and shown in Fig. 6.6, and its height is h. Let us calculate the moment of inertia I_A about the axis shown, which is through the corner A and perpendicular to the face $ABCD$. Consider a rectangular strip of cross-section $A'B'C'D'$ and thickness δx. The mass of this strip is $(M\delta x/h)$, where M is the mass of the whole cuboid. By (6.12) the contribution δI_A of the strip to I_A satisfies

$$\delta I_A = \frac{1}{3}\left(\frac{M\delta x}{h}\right)(b^2 + l^2)$$

$$\Rightarrow \quad I_A = \frac{1}{3}\left(\frac{M}{h}\right)(b^2 + l^2)\int_O^h \mathrm{d}x = \frac{1}{3}M(b^2 + l^2).$$

In the next example we consider the moment of inertia of a sphere.

Example 4 Determine the moment of inertia of a uniform sphere of radius R and mass M about a diameter.

Let O be the centre of the sphere, and let us find the moment of inertia I_A about the diameter through A, as shown in Fig. 6.8. Consider a circular disc whose plane is perpendicular to OA, which has thickness δx and centre C, where $OC = x$. This disc has radius $CB = \sqrt{(OB^2 - OC^2)} = \sqrt{(R^2 - x^2)}$. Hence its volume is $\pi(R^2 - x^2)\delta x$, so that its mass is

$$M\frac{\pi(R^2 - x^2)\delta x}{\frac{4}{3}\pi R^3} = \frac{3M}{4R^3}(R^2 - x^2)\delta x.$$

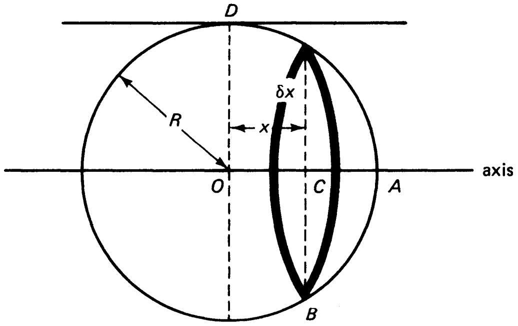


Fig. 6.8

From (6.9), it follows that the contribution δI_A of this disc to I_A satisfies

$$\delta I_A = \frac{1}{2}\frac{3M}{4R^3}(R^2 - x^2)\delta x\,.(CB)^2 = \frac{3M}{8R^3}(R^2 - x^2)^2\delta x$$

$$\Rightarrow \quad I_A = \frac{3M}{8R^3}\int_{-R}^{R}(R^2 - x^2)^2\,\mathrm{d}x$$

$$= \frac{3M}{8R^3}\int_{-R}^{R}(R^4 - 2R^2x^2 + x^4)\,\mathrm{d}x$$

$$= \frac{2}{5}MR^2. \tag{6.13}$$

We can apply the theorem of parallel axes (6.8) to solid bodies. Thus, for the sphere just considered, the moment of inertia I_D about an axis through a point D on the surface of the sphere and tangential to the sphere satisfies, by (6.8),

$$I_D = I_A + MR^2 = \tfrac{7}{5}MR^2.$$

However, the theorem of perpendicular axes (6.10) applies only to laminae and cannot be applied to solid bodies.

All the examples show that, for all points O and for all directions of the axis through O, the moment of inertia I_O is given by a formula of the form

$$I_O = Mk^2, \tag{6.14}$$

where M is the mass of the body and k is a constant with dimension L. The term *radius of gyration* is used for k; k depends (in general) on the axis as well as on its position. Thus, for the rigid rod of length $2l$ considered in Example 1 (p. 67), the value of k is $l\surd(4/3)$ for an axis perpendicular to the rod through one end and is $l\surd(1/3)$ for a parallel axis through the centre of the rod.

6.3 Some simple cases of rotational motion

We are now in a position to use the equation of rotational motion in some simple examples. This equation is (6.1) and is repeated here for convenience.

$$I_O\ddot{\theta} = L.$$

In this equation L is the moment of the external forces about the fixed axis in the anti-clockwise sense. We begin with an example in which L is constant.

Example 5 Find the moment of inertia of a uniform square lamina $ABCD$ of mass M and side a about the diagonal AC, as shown in Fig. 6.9.

The lamina is able to rotate about a fixed, vertical axis coinciding with AC and is given an initial angular speed ω_O. It is acted on by a constant driving couple L. Show that when the lamina has turned through an angle θ its angular speed is $\sqrt{[\omega_O{}^2 + (24\theta L/Ma^2)]}$.

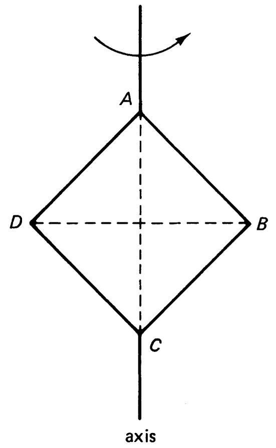


Fig. 6.9

From (6.11), the moment of inertia about an axis perpendicular to the lamina through its centre O is

$$\tfrac{1}{12}M(a^2 + a^2) = \tfrac{1}{6}Ma^2.$$

By symmetry its moment of inertia I about AC is equal to its moment of inertia about BD. Hence, by (6.10),

$$2I = \tfrac{1}{6}Ma^2 \quad \Rightarrow \quad I = \tfrac{1}{12}Ma^2.$$

Thus (6.1) becomes

$$\tfrac{1}{12}Ma^2\ddot{\theta} = L.$$

Let

$$\ddot{\theta} = \alpha \quad \text{where } \alpha = (12L/Ma^2). \tag{6.15}$$

By integrating once, $\dot{\theta} = \alpha t +$ constant. Since $\dot{\theta} = \omega_O$ when $t = 0$, the value of the constant is ω_O. Thus

$$\dot{\theta} = \alpha t + \omega_O. \tag{6.16}$$

We can integrate once more to obtain $\theta = \frac{1}{2}\alpha t^2 + \omega_O t +$ constant. If we measure θ from the position at $t = 0$ (as required by the question) the constant is zero. Hence

$$\theta = \tfrac{1}{2}\alpha t^2 + \omega_O t. \tag{6.17}$$

Equation (6.17) can be written as

$$\theta = \frac{1}{2\alpha}(\alpha^2 t^2 + 2\alpha\omega_O t) = \frac{1}{2\alpha}(\alpha t + \omega_O)^2 - \frac{\omega_O{}^2}{2\alpha}.$$

Hence, using (6.16),

$$\theta = \frac{\dot{\theta}^2}{2\alpha} - \frac{\omega_O{}^2}{2\alpha} \quad \Rightarrow \quad \dot{\theta}^2 = \omega_O{}^2 + 2\alpha\theta. \tag{6.18}$$

This gives the required result when α is replaced by $(12L/Ma^2)$.

The motion in the preceding example is one with constant *angular acceleration* α, and in terms of formal mathematical structure, but not of physical significance, the governing equation (6.15) is the same as that governing motion in a straight line with constant acceleration. Thus, equations (6.16), (6.17) and (6.18) have direct analogues with equations applying to motion in a straight line with constant acceleration.

The second example that we shall consider is one in which L in (6.1) is not constant but depends on θ.

Example 6 A rigid body whose centre of mass is G is free to rotate about a fixed horizontal axis l. Show that the body can perform small oscillations whose period T is given approximately by

$$T = 2\pi\sqrt{\left(\frac{h^2 + k^2}{gh}\right)},$$

where h is the distance of G from l, and k is the radius of gyration of the body about an axis through G parallel to l. Find the minimum value of T as h varies.

The situation is illustrated in Fig. 6.10, where O is the point on the axis l which lies in the plane through G perpendicular to l. We define θ to be the angle made by OG with the downward vertical. From the theorem of parallel axes, the moment of inertia I_O of the body about l is given by $I_O = I_G + Mh^2 = M(h^2 + k^2)$, since we are given that $Mk^2 = I_G$, the moment of inertia of the body about an axis through G parallel to l. Hence (6.1) becomes

$$M(h^2 + k^2)\ddot{\theta} = L,$$

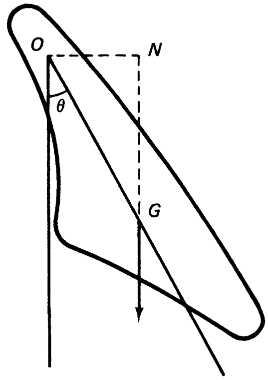


Fig. 6.10

where L is the moment of the external forces about l. The only external forces acting on the body are its weight, acting vertically downwards through G, and the force exerted by the axis on the body—but this latter force acts through l and so does not contribute to L. It is evident that the weight has a clockwise moment about l of magnitude $Mg\,.\,ON = Mgh\sin\theta$. Hence $L = -Mgh\sin\theta$ and the above equation becomes

$$M(h^2 + k^2)\ddot{\theta} = -Mgh\sin\theta \quad \Rightarrow \quad \ddot{\theta} = -n^2\sin\theta, \qquad (6.19)$$

where the constant n^2 is given by

$$n^2 = \frac{gh}{h^2 + k^2}. \qquad (6.20)$$

Common-sense tells us that the body can rest without moving with G vertically below O, that is with $\theta = 0$ for all t. This belief is consistent with (6.19) for when $\theta = 0$ for all t, then $\ddot{\theta} = \sin\theta = 0$ for all t and (6.19) is (trivially) satisfied.

Now suppose that while at rest, in the position just described, the body is *slightly* disturbed, for example by a blow. Experience would then suggest that it oscillates about the downward vertical with θ always small. When θ is small, $\sin\theta \approx \theta$ and (6.19) can be approximated by

$$\ddot{\theta} = -n^2\theta. \qquad (6.21)$$

As seen already in §1.5, p. 11, equation (6.21) is the equation of simple harmonic motion which does indeed produce oscillations of period T, where

$$T = \frac{2\pi}{n} = 2\pi\sqrt{\left(\frac{h^2 + k^2}{gh}\right)},$$

using (6.20). This is the required value of T.

As h varies, the minimum value of T (if any) occurs for that value of h which makes $f(h)$, where $f(h) = (h^2 + k^2)/gh$, a minimum. By using the quotient rule,

we find

$$f'(h) = \frac{gh \,.\, 2h - (h^2 + k^2)g}{g^2h^2} = \frac{1}{gh^2}(h^2 - k^2).$$

Thus $f'(h) = 0$ when $h = k$. We can show that $h = k$ gives a minimum value of $f(h)$ either by showing that $f''(h) > 0$ when $h = k$, or by noting, by inspection of the formula for $f'(h)$, that $f'(h) < 0$ for h slightly less than k and that $f'(h) > 0$ for h slightly greater than k. Therefore the minimum value T_{min} of T is given by

$$T_{min} = 2\pi\sqrt{\left(\frac{k^2 + k^2}{gk}\right)} = 2\pi\sqrt{\left(\frac{2k}{g}\right)}.$$

Just as we noted that the work in Example 5 was mathematically similar to work done earlier on the motion of a particle, we now note that the work in Example 6 bears a striking similarity to earlier work in this book. In this case the similarity is with the motion of a particle in a vertical circle (§1.4) and a simple pendulum (§1.5). Indeed (6.19) and (6.21) are, except for unimportant changes in the notation used for the constants, the same as the first equations of (1.22) and (1.27) respectively. This similarity is also recognized by terminology. When a rigid body is performing small oscillations, as in Example 6, it is said to be a *compound pendulum*.

6.4 Energy considerations for a rigid body rotating about a fixed axis

The analogy just noted between the rotation of a rigid body about a fixed axis and the motion of a particle extends, in many respects, to the laws of work and energy considered in Chapter 1.

Let P be a typical particle of the rigid body and suppose that P has mass m. As noted in §5.4, P moves along a circle whose centre is O, the point where the axis of rotation meets the plane through P perpendicular to the axis. Following the notation used in §5.4 and the present chapter, and shown in Fig. 5.7, let the radius of this circle be r. From the kinematics of circular motion, the velocity $\mathbf{v}$ of P satisfies $\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}}$. Hence, from the definition in (1.36), the kinetic energy T_P of P is given by

$$T_P = \tfrac{1}{2}m\mathrm{v}^2 = \tfrac{1}{2}mr^2\dot{\theta}^2. \tag{6.22}$$

The total kinetic energy T of the body is ΣT_P, so that

$$T = \sum \tfrac{1}{2}mr^2\dot{\theta}^2 = \tfrac{1}{2}I_O\dot{\theta}^2, \tag{6.23}$$

where we have used the fact that all particles of the body have the same angular speed $\dot{\theta}$, and I_O is defined by (6.2). Since I_O does not change when the body is moving,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \dot{T} = I_O\ddot{\theta}\dot{\theta} = L\dot{\theta}, \tag{6.24}$$

using (6.1). Thus the rate of increase of the kinetic energy is equal to $L\dot{\theta}$. By analogy with the work in Chapter 1, it is natural to define $L\dot{\theta}$ as the power. This can be taken further. By definition

$$\dot{\theta} = \lim_{\delta t \to 0} \left(\frac{\delta\theta}{\delta t}\right),$$

where $\delta\theta$ is the angle turned through by OP (or any line fixed in the body and perpendicular to the axis) in the small time interval from t to $t + \delta t$. Hence the power $L\dot{\theta}$ satisfies

$$L\dot{\theta} = \lim_{\delta t \to 0} \left(\frac{L\delta\theta}{\delta t}\right). \tag{6.25}$$

Earlier, in §1.7, we showed that, for the motion of a particle, the power was the rate of doing work. We now see from (6.25) that, provided we define $L\delta\theta$ as the *work done* by the external forces as the body rotates through an angle $\delta\theta$, we can retain this interpretation.

For the special case when L is constant, it now follows that the work done when the body turns through an angle θ, which is not small, is equal to $L\theta$. Furthermore, in this case, $\dfrac{\mathrm{d}}{\mathrm{d}t}(L\theta) = L\dot{\theta}$ so that, by integrating (6.24) with respect to t from t_1 to t_2, we obtain

$$\Big[T\Big]_{t_1}^{t_2} = \Big[L\theta\Big]_{t_1}^{t_2} \quad \Rightarrow \quad \left[\frac{1}{2}I_O\dot{\theta}^2\right]_{t_1}^{t_2} = L(\theta_2 - \theta_1). \tag{6.26}$$

The similarity to (1.4) is obvious and remarkable.

More generally, when L is not constant, we can proceed as in §1.2. There we noted that, for a particle moving in a straight line under a force of magnitude $F(x)$, it was possible and useful to define a potential energy $\Phi(x)$ by (1.12), namely

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = -F.$$

Suppose that now, by analogy, we define a *potential energy* $\Phi(\theta)$ by the equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\theta} = -L, \tag{6.27}$$

where we can choose for Φ any quantity satisfying (6.27). Then

$$L\dot{\theta} = -\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}\cdot\dot{\theta} = -\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t},$$

and so (6.24) becomes

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(T + \Phi) = 0.$$

Hence $(T + \Phi)$ is constant or, in words, we recover (1.9), the principle of conservation of energy:

$$\text{'kinetic energy + potential energy = constant'}. \qquad (6.28)$$

Let us consider two special cases very briefly. First, when L is constant, we can, according to (6.27), take $\Phi = -L\theta$. Hence $(T + \Phi) = \frac{1}{2}I_O\dot{\theta}^2 - L\theta = \text{constant}$. If $\theta = \theta_1$ when $t = t_1$ and $\theta = \theta_2$ when $t = t_2$, we obtain

$$\left[\frac{1}{2}I_O\dot{\theta}^2 - L\theta\right]_{t_1}^{t_2} = 0 \quad \Rightarrow \quad \left[\frac{1}{2}I_O\dot{\theta}^2\right]_{t_1}^{t_2} = L(\theta_2 - \theta_1),$$

which is, once more, equation (6.26). The reader should also confirm that equation (6.18) is an immediate consequence of the same result.

Secondly, consider the motion studied in Example 6, p. 75, when $L = -Mgh\sin\theta$, so that

$$\frac{d\Phi}{d\theta} = +Mgh\sin\theta \quad \Rightarrow \quad \Phi = Mgh(1 - \cos\theta), \qquad (6.29)$$

where we have, for convenience, chosen the constant of integration so that $\Phi = 0$ when $\theta = 0$. For this motion $I_O = M(h^2 + k^2)$ and (6.28) takes the form

$$\tfrac{1}{2}M(h^2 + k^2)\dot{\theta}^2 + Mgh(1 - \cos\theta) = \text{constant}. \qquad (6.30)$$

The constant is equal to the value of $Mgh(1 - \cos\theta)$ when $\dot{\theta} = 0$. Let this occur when $\theta = \beta$. Hence the constant is equal to $Mgh(1 - \cos\beta)$ so that

$$\tfrac{1}{2}M(h^2 + k^2)\dot{\theta}^2 + Mgh(1 - \cos\theta) = Mgh(1 - \cos\beta)$$
$$\Rightarrow \quad \tfrac{1}{2}\dot{\theta}^2 + n^2(1 - \cos\theta) = n^2(1 - \cos\beta), \qquad (6.31)$$

where n is defined in (6.20). This equation can be used to investigate, as in §1.4, the motion of the rigid body. For the particular case studied in Example 6, θ was always small, so that $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\cos\beta \approx 1 - \frac{1}{2}\beta^2$. Hence (6.31) becomes approximately

$$\tfrac{1}{2}\dot{\theta}^2 + n^2(\tfrac{1}{2}\theta^2) = n^2(\tfrac{1}{2}\beta^2) \quad \Rightarrow \quad \dot{\theta}^2 + n^2\theta^2 = n^2\beta^2,$$

which, from the work in §1.3, is the energy equation for simple harmonic motion, consistent with the conclusions of Example 6 reached by a different method.

Exercise 6

1 A uniform, circular plate of radius a and mass $2M$ has centre O. It is bisected by a cut along a diameter AB. One of the resulting semi-circular plates is suspended so that it is free to turn in a vertical plane about a smooth horizontal axis through O. Show that the centre of mass of the plate is at a distance $4a/3\pi$ from O, and that its moment of inertia about the axis through O is $\frac{1}{2}Ma^2$. The plate is held at rest with AB vertical and then released. Find its angular speed when it has turned through an angle θ.

2 The vertices O, S and T of a uniform, triangular lamina of mass M have cartesian coordinates $(0, 0)$, $(a, 2a)$ and $(a, -a)$ respectively. Draw a sketch of the lamina, mark on it the point G with coordinates $(\frac{2}{3}a, \frac{1}{3}a)$, and state the physical significance of G. By integration find the moments of inertia of the lamina about (a) the x-axis, and (b) the y-axis. Hence find the moments of inertia about (c) the axis through O perpendicular to the lamina, and (d) the axis through G perpendicular to the lamina.

3 Two equal rigid uniform rods AB and BC, each of mass m and length l, are rigidly fastened at B so that they are perpendicular. The resultant body lies on a smooth, horizontal table, with A, B and C in contact with the table, and can rotate about a fixed vertical axis through A. Find the moment of inertia of the body about this axis. The body is rotating with constant angular speed ω when a constant resisting couple is applied so that the angular speed is reduced to zero after exactly one revolution. Find the moment of the couple, and the components, along and perpendicular to AB, of the acceleration of B immediately after the couple is applied.

4 A uniform wire of length $8l$ and mass $4m$ is bent into the form of a square $PQRS$ of side $2l$. The square is placed on a smooth, horizontal table and smoothly hinged at P so that it is free to rotate about a vertical axis through P. When the wire is rotating on the table with angular speed Ω about P, it is brought to rest in time T by a constant couple of moment G. In this time the wire completes n revolutions. Find T and n in terms of m, l, G and Ω.

5 A uniform circular lamina of centre O, mass m and radius a is free to rotate in a vertical plane about a horizontal axis perpendicular to its plane through a point A, where $OA = \frac{1}{2}a$. It is held at rest with OA horizontal and then released. Show that when OA has turned through an angle θ, the speed of O is $\sqrt{(\frac{1}{3}ag\sin\theta)}$.

6 A wheel has a horizontal cylindrical axle of radius a. The system of wheel and axle has mass M and radius of gyration k about its axis and rotates without friction. A light, thin, inextensible string is wound round the axle and has one end attached to a point on the axle. To the other end of the string is attached a particle P of mass m which hangs vertically. Find the downward acceleration of P when the system is released from rest.

7 Show that the moment of inertia of a uniform, solid, right circular cone of mass m, height h and base radius a about a line through its vertex perpendicular to its axis is

$$\frac{3m}{20}(a^2 + 4h^2).$$

The cone is free to rotate about a smooth horizontal axis through its vertex. Find the period of small oscillations under gravity.

8 A uniform rectangular lamina $ABCD$, in which $AD = 2l$, is smoothly hinged along XY, where XY is in the plane of the lamina and is parallel to AB at a distance x from the centre of gravity of the lamina. When the lamina performs small oscillations about XY, which is fixed and horizontal, the period is T. Show that

$$T = 2\pi\sqrt{\left(\frac{l^2 + 3x^2}{3gx}\right)},$$

and find the value of x for which T is least.

Answers

Exercise 1

1 (a) 29400 J, (b) 45000 J; 74·4 kW.
2 50000 N, 20000 N.
3 1400 N, 58800 J, 5880 W, 18 m.
4 P/v—tractive force due to working at constant power, $-kv^2$—air resistance; $v_0 = (P/k)^{1/3}$.
5 2 J.
6 $\frac{3}{2}\sqrt{(ga)}$, a.
7 They stay in.
8 1·74 s, 6·3°.
9 $\sqrt{(2ga^2/x)}$.
10 $4Fa$.

Exercise 2

1 10 kgms^{-1}, 9 kgms^{-1}.
2 2568 J.
3 $\sqrt{(2kE/mk + m)}$.
6 A: $(1/3)U(1 - 2e)$, B: $(1/9)U(1 - e - 2e^2)$, C: $(1/9)U(1 + 2e + e^2)$; two collisions only.
8 Mass M, $2V/g$; mass $3M$, $(1 + \sqrt{17})V/g$.
9 1·58 m.
10 $(2m/5)\sqrt{(2ga)}$; $m\sqrt{(2ga)}$.

Exercise 3

1 $(-\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k})$ m.
2 (31/22) cm from base of bottom disc.
5 $A = 4/3$; $(\bar{x}, \bar{y}) = (4/5, 16/35)$.
6 $11a/14$.
7 $(R \sin \alpha)/\alpha$ from centre along radius of symmetry.
8 $A = 1$; $(\bar{x}, \bar{y}) = (\frac{1}{2}\pi - 1, \frac{1}{8}\pi)$.

Exercise 4

1 $17l/6$.
3 $3M/5$.
4 $\mu = (\sqrt{3} - 1)/(\sqrt{3} + 1) = 2 - \sqrt{3}$.
5 $\pi/4$; half-way up.
7 $5Mg/4$.
11 (a) $4a$; (b) $3mga/2$; (c) horizontal—$3mga$, vertical—$13mga/2$; (d) $\frac{1}{2}$.
12 $W \tan(\gamma/2)$.
13 $\sigma = \mu \tan\alpha + (\mu \tan\alpha - \frac{1}{2})/\lambda$.

Exercise 5

1 2 N, 3 m.
2 $(2K\sqrt{3})/(3a)$.
3 (b) $\sqrt{(205)}$ N.
4 $4M$.
5 $\alpha = 1$, $\beta = 4$.
6 $L = \mu Mga \sec\beta/(\mu^2 + 1)$.

Exercise 6

1 $\sqrt{[(16g \sin\theta)/(3\pi a)]}$.
2 G is the centre of mass of the lamina; (a) $\frac{1}{2}Ma^2$, (b) $\frac{1}{2}Ma^2$, (c) Ma^2, (d) $\frac{4}{9}Ma^2$.
3 $5ml^2/3$; $5ml^2\omega^2/12\pi$; $l\omega^2$ along AB (towards A), $l\omega^2/4\pi$ along BC.
4 $T = 40ml^2\Omega/3G$, $n = 10ml^2\Omega^2/3\pi G$.
6 $mga^2/(ma^2 + Mk^2)$.
7 $2\pi\sqrt{[(a^2 + 4h^2)/5gh]}$.
8 $l/\sqrt{3}$.

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